A duality Theorem:

X, Y complete, separable, locally cpt metric space.

$c: X \times Y \to \mathbb{R} \cup \{+\infty\}$. l.s.c, bdd from below.

$\mu \in P(X)$, $\nu \in P(Y)$.

(1). $\pi \in \Pi(\mu, \nu)$. $I[\pi] = \int_{X \times Y} c(x, y) \, d\pi(x, y)$.

(2) $(\varphi, \psi) \in C_b(X) \times C_b(Y)$. $\varphi(x) + \psi(y) \le c(x, y) \rightsquigarrow \Phi_c$.

$$J(\varphi, \psi) = \int_X \varphi \, d\mu + \int_Y \psi \, d\nu.$$

Then: $\inf_{\pi \in \Pi(\mu, \nu)} I[\pi] = \sup_{(\varphi, \psi) \in \Phi_c} J(\varphi, \psi)$.

Now a rigorous proof.

LHS $\ge$ RHS obvious

Non-obvious

LHS $\le$ RHS.

Just consider the special case:

X, Y cpt.

c is continuous.

Choose $(\varphi_n, \psi_n) \in \Phi_c$ s.t.

$$J(\varphi_n, \psi_n) \to \sup_{(\varphi, \psi) \in \Phi_c} J(\varphi, \psi)$$

Claim 1: It is possible to choose (φ_n, ψ_n) to be c-concave. i.e
we can chose (φ_n, ψ_n) s.t

$$\varphi_n(x) = \inf_{y \in Y} \left(C(x, y) - \psi_n(y) \right) \qquad \varphi_n(x_0) = 0$$

$$\psi_n(y) = \inf_{x \in X} \left(C(x, y) - \varphi_n(x) \right).$$

Then. $\varphi_n(x)$ & $\psi_n(y)$ is equi-cont. Moreover $J(\varphi_n + a, \psi_n - a)$
$= J(\varphi_n, \psi_n)$

Assume the claim 1: $\varphi_n \longrightarrow \varphi_\infty$. $\psi_n \longrightarrow \psi_\infty$ uniformly.

$$J(\varphi_n, \psi_n) \longrightarrow J(\varphi_\infty, \psi_\infty) \qquad \varphi_\infty(x) + \psi_\infty(y) \le C(x, y)$$

wish to show $\sup_{(\varphi,\psi) \in \Phi_c} J(\varphi, \psi) = J(\varphi_\infty, \psi_\infty) = \inf_{\pi \in \Pi(\mu, \nu)} I[\pi] = \int_{X \times Y} c(x, y) \, d\pi_0$.

Claim 2: $\varphi_\infty(x) + \psi_\infty(y) = C(x, y)$ π_0-a.e. Assume this claim 2, we are done!

$$\int_{X \times Y} c(x, y) \, d\pi_0 = \int_{X \times Y} (\varphi_\infty(x) + \psi_\infty(y)) \, d\pi_0(x, y) = \int_X \varphi_\infty(x) \, d\mu + \int_Y \psi_\infty(y) \, d\nu.$$

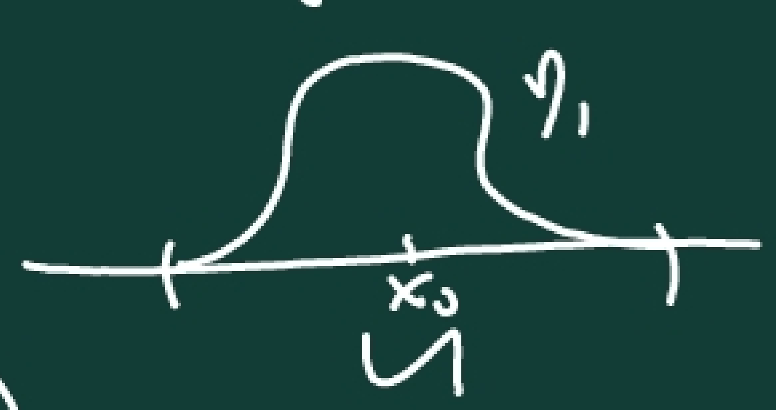
proofs of claim 1 & 2.

<u>Claim 2</u>: $\varphi_\infty(x)+\psi_\infty(y)=c(x,y)$. π_0-a.e. $\pi_0\in\Pi(\mu,\nu)$

If not, let $(x_0,y_0)\in \mathrm{supp}(\pi_0)$. s.t $\varphi_\infty(x_0)+\psi_\infty(y_0)<c(x_0,y_0)$

$\varphi_\infty, \psi_\infty$ cont. $\exists\,(U, V)\ni(x_0,y_0)$. s.t $\varphi_\infty(x)+\psi_\infty(y)<c(x,y)-\delta$.

$\eta_1\in C_c(U)$. $\eta_2\in C_c(V)$. $\eta_1,\eta_2\geq 0$.



$\big(\varphi_\infty(x)+\delta'\eta_1(x)\big)+\big(\psi_\infty(y)+\delta'\eta_2(y)\big)\leq c(x,y)$, $(x,y)\in X\times Y$.

$J(\varphi_\infty+\delta'\eta_1, \psi_\infty+\delta'\eta_2)>J(\varphi_\infty,\psi_\infty)$. Contradiction.

<u>Claim 1</u>: Why you can choose (φ_n,ψ_n) to be c-concave.

Since $\varphi_n(x)+\psi_n(y)\leq c(x,y)$

$\psi_n(y) \le c(x,y) - \varphi_n(x)$. So if we choose $\tilde{\psi}_n(y) = \inf_{x\in X}(c(x,y)-\varphi_n(x))$.

$\psi_n(y) \le \tilde{\psi}_n(y)$. $\varphi_n(x) + \tilde{\psi}_n(y) \le c(x,y)$.

$$\tilde{\varphi}_n(x) = \inf_{y\in Y}(c(x,y) - \tilde{\psi}_n(y)).$$

$\varphi_n \le \tilde{\varphi}_n$, $\psi_n \le \tilde{\psi}_n$. $\tilde{\varphi}_n(x) + \tilde{\psi}_n(y) \le c(x,y)$.

what if you do it again... $\hat{\psi}_n(y) = \inf_{x\in X}(c(x,y) - \tilde{\varphi}_n(x))$

then you actually have $\hat{\psi}_n(y) = \tilde{\psi}_n(y)$

WLOG. can assume $\varphi_n(x_0) = 0$. $\tilde{\psi}_n(y) \le c(x_0,y) - \varphi_n(x_0)$

$$= c(x_0,y) \rightsquigarrow \text{upper bound.}$$

$\Rightarrow \tilde{\varphi}_n \ge -c$

Now we specialize to $c(x,y) = \frac{1}{2}|x-y|^2$.

Thm: Let $\mu, \nu \in P(\mathbb{R}^n)$ with finite 2nd order moment. ($\int |x|^2 d\mu(x) < \infty$, $\int |y|^2 d\nu(y) < \infty$).

(1). $\pi \in \Pi(\mu, \nu)$ minimizes $I[\pi]$ iff $\exists$ convex function φ. s.t for π-a.e (x, y). $y \in \partial\varphi(x)$. $\rightsquigarrow \varphi(x') \geq \varphi(x) + y\cdot(x'-x)$ $\forall x'$.

(2). (Brenier). If $\mu \ll dm$. then $\exists!$ optimal π. s.t $\pi = (id \times D\varphi)_{\#}\mu$. Moreover, $D\varphi$ is uniquely determined. & solves $\inf_{T_{\#}\mu = \nu} \int_{\mathbb{R}^n} |x - Tx|^2 d\mu(x)$

(3). If. $\mu, \nu \ll dm$. $\exists \varphi, \varphi^*$ convex. s.t $D\varphi$ is the optimal map $\mu \to \nu$. $D\varphi^*$ optimal map $\nu \to \mu$. & $D\varphi \circ D\varphi^* = id(y)$ for ν-a.e y. $D\varphi^* \circ D\varphi(x) = x$ μ-a.e x.

Some consequences of the thm:

Isoperimetric ineq. $E \subseteq \mathbb{R}^n$ bdd domain with smooth bdry. then

$$\frac{(P(E))^{\frac{1}{n-1}}}{vol(E)^{\frac{1}{n}}} \geq \frac{(P(B_1))^{\frac{1}{n-1}}}{(vol(B_1))^{\frac{1}{n}}}. \quad \text{"="} \text{ iff } E = B.$$

proof: WLOG, can assume $vol(E) = vol(B_1)$. Need to show

$P(E) \geq P(B_1)$. Then we consider $m|_E$ & $m|_{B_1}$.

Thm $\Longrightarrow$ $\exists$ convex function φ. s.t. $\nabla\varphi_{\#}(m|_E) = m|_{B_1}$.

Using change of variable formula. $\det D^2\varphi = 1$ on E.

$$vol(B_1) = vol(E) = \int_E \det{}^{\frac{1}{n}}(D^2\varphi)\,dx \leq \int_E \frac{1}{n}\sum_i \lambda_i(x)\,dx = \int_E \frac{1}{n}\Delta\varphi(x)\,dx$$

"=" $\lambda_i(x) = 1$ $i = 1, 2, \ldots, n$.

$$= \int_{\partial E} \frac{1}{n} \nabla\varphi \cdot n \, d\sigma.$$

$\nabla\varphi(E) \subseteq B_1$. $|\nabla\varphi(x)| \le 1$. , $\int_{\partial E} \nabla\varphi \cdot n \, d\sigma \le \sigma(\partial E)$.

$vol(B_1) \le \frac{1}{n} P(E)$. $\quad n\, vol(B_1) = P(B_1)$

(Brenier's polar factorization)

Let $\Omega \subseteq \mathbb{R}^n$, $\infty > m(\Omega) > 0$. Let $h : \Omega \to \mathbb{R}^n$ be in L^2 with the non-degenerate condition: $m(h^{-1}(A)) = 0$, $\forall A \subseteq \mathbb{R}^n$ with measure 0. Then

$$h = \nabla\varphi \circ S. \quad , \text{where } \varphi : \mathbb{R}^n \to \mathbb{R} \text{ Convex},$$

$S : \Omega \to \Omega$ measure preserving

$m(S(A)) = m(A)$. $\forall A \subseteq \Omega$.

Moreover, S is the projection of h onto $\mathcal{S}(\Omega)$, i.e

$$\|h - S\|_{L^2} = \inf_{\sigma \in \mathcal{S}(\Omega)} \|h - \sigma\|_{L^2}.$$

$\mathcal{S}(\Omega) \to \sigma : \Omega \to \Omega$. measure preserving