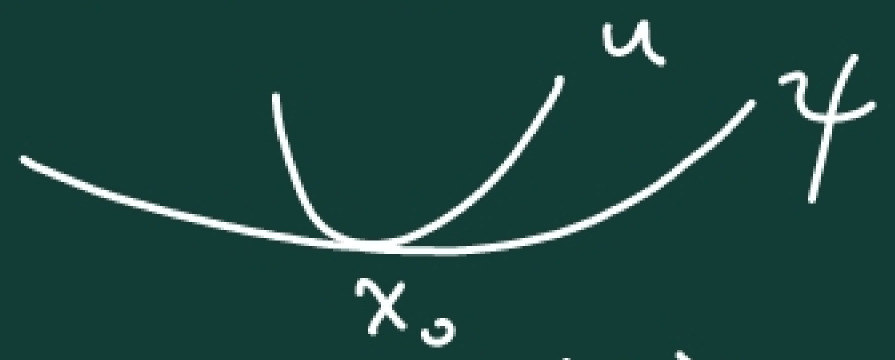
3 types of weak solutions to $\det D^2 u = f(x, u(x), \nabla u(x))$

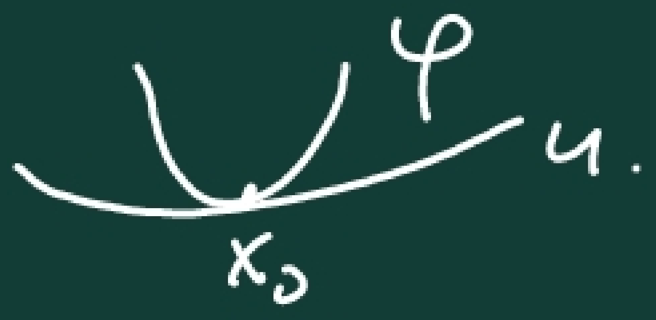
(1) Alexandrov solution: Consider $\mu E = m(\partial u(E)) \rightsquigarrow$ Hessian measure of u.

$\mu = f(x, u(x), Du(x))\,dx$ as equality of measures.

(2) Viscosity solution: (1) Let $\psi \in C^2$, convex, touch u from below at x_0



$\det D^2\psi(x_0) \leq f(x_0, \psi(x_0), \nabla\psi(x_0))$

(2) 

$\varphi \in C^2$, convex, touch u from above at x_0, $\det D^2\varphi(x_0) \geq f(x_0, \varphi(x_0), \nabla\varphi(x_0))$

(3) (Brenier solution) $\det D^2 u = \dfrac{f(x)}{g(\nabla u(x))}$

$\nabla u_{\#}(f(x)\,dx) = g(y)\,dy$ i.e $\forall A \subseteq \mathbb{R}^n$: $\displaystyle\int_A g(y)\,dy = \int_{\nabla\varphi^{-1}(A)} f(x)\,dx$.

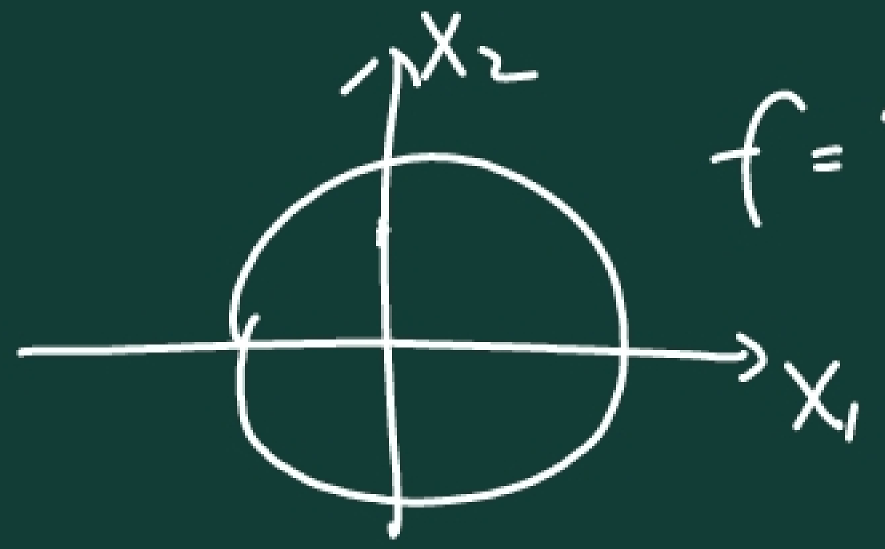
Q: When is Brenier solution a Alexandrov solution?

Brenier soln is not necessarily Alexandrov!

e.g. $\varphi(x) = \frac{1}{2}(x_1^2 + x_2^2) + |x_1|$.

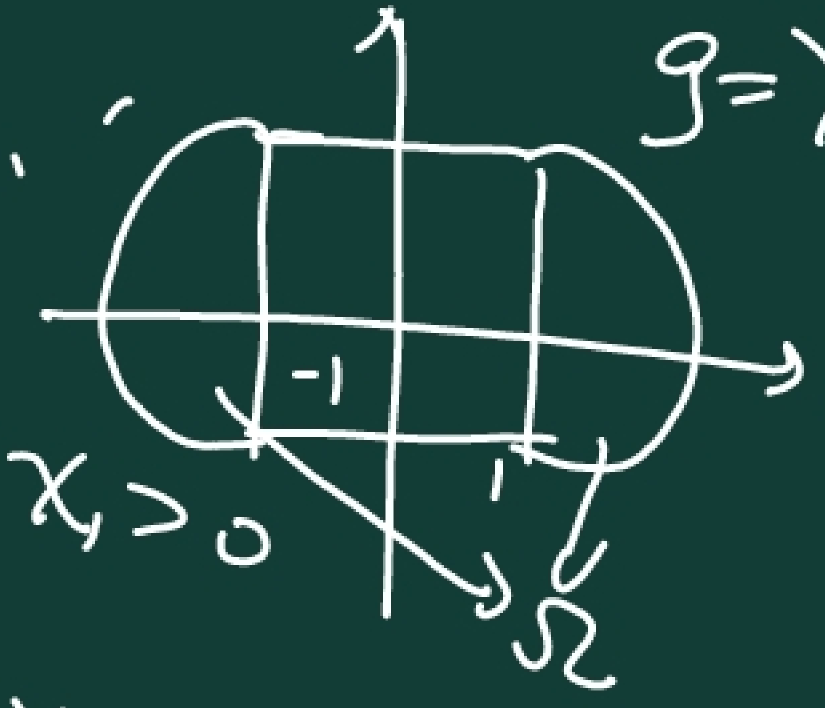
Claim: φ is Brenier solution to $\det D^2\varphi(x) = 1$, $D\varphi_\#(f\,dx) = g\,dy$.

where



$f = \chi_{B_1} dx$.

$$D\varphi = \begin{cases} (x_1, x_2) + e_1, & x_1 > 0 \\ (x_1, x_2) - e_1, & x_1 < 0 \end{cases}$$



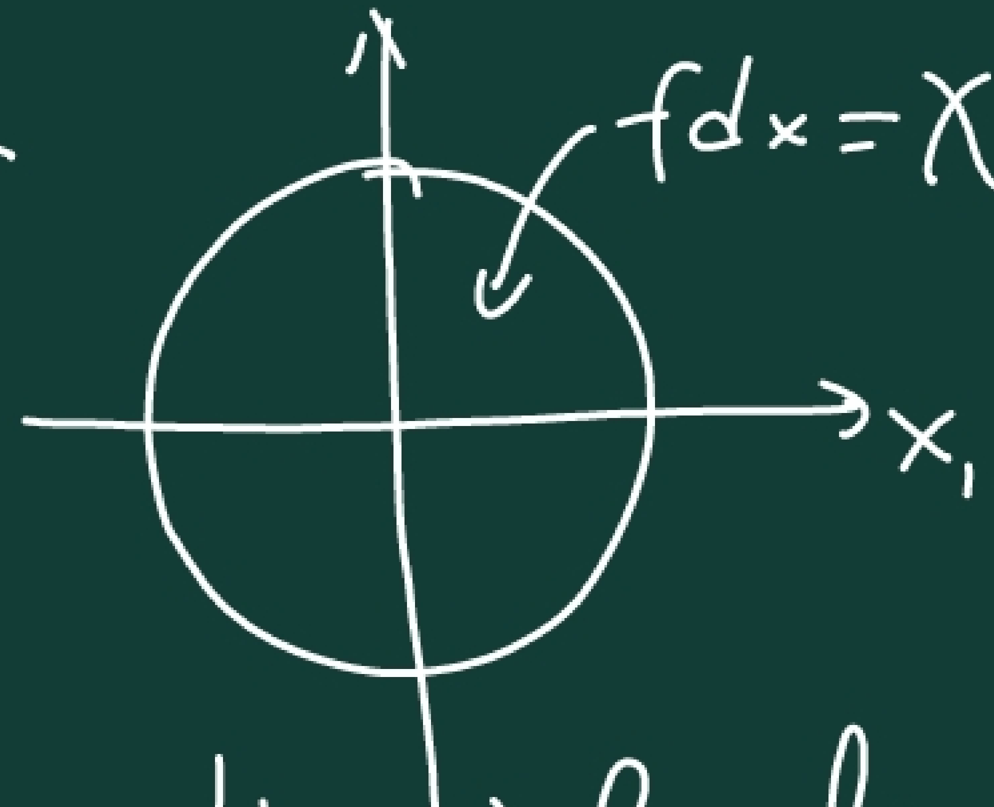
$g = \chi_\Omega dy$

But φ is not Alexandrov, since $\det D^2_H \varphi = \chi_{B_1} + \mathcal{H}^1\big|_{\{x_1 = 0\}}$.

In this example, the support of target measure is non-connected.

Q: If the support of target measure is connected, is  Brenier solution an Alexandrov soln?

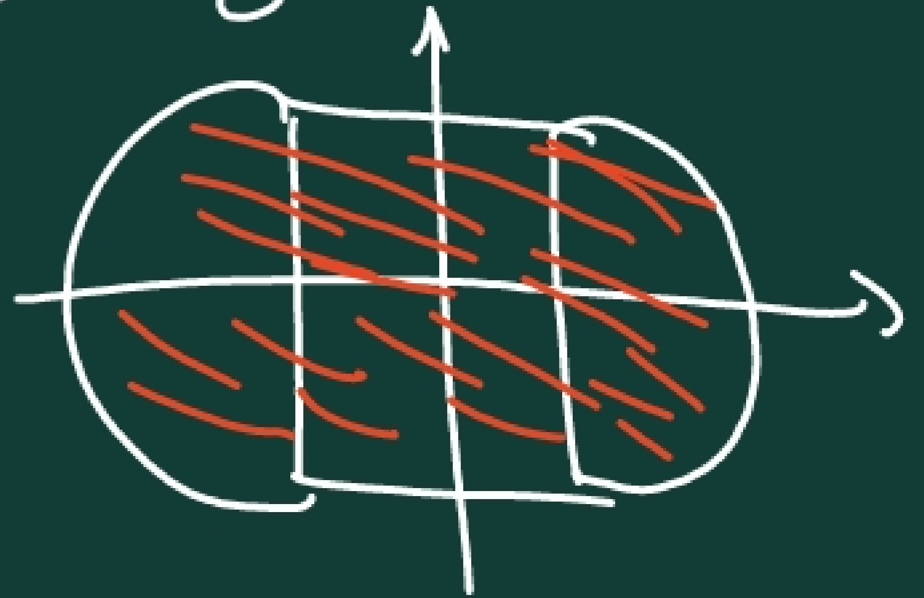
Ans: NO! Consider



$\xrightarrow{\nabla \varphi_\epsilon}$

As $\epsilon \to 0$. $\underbrace{\det D_H^2 \varphi}_{\text{total mass}} \le \liminf_{\epsilon \to 0} \det D_H^2 \varphi_\epsilon$.

$\partial\varphi\{x_0\}$



$\Rightarrow$ total mass of $\det D_H^2 \varphi_\epsilon > |B_1|$.
$\epsilon \to 0$.

Thm (Caffarelli). Let φ be the Brenier soln to $\det D^2\varphi = \frac{f(x)}{g(D\varphi(x))}$. Assume $\{g>0\}$ is convex. Then φ solves the equation in the Alexandrov sense.

<u>plan</u>: $\det D_H^2 \varphi = h(x)dx + \mu$. $\mu \perp m$.

we need to show: (1) $h(x) = \frac{f(x)}{g(D\varphi(x))}$ a.e x.

(2). $\mu = 0$.

<u>prop</u>: Let $\nabla\varphi_\#(f(x)dx)=g(y)dy$. $\varphi:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$. $\Omega=\mathrm{Int}\{\varphi<+\infty\}$.

$\det D_A^2\varphi=$ det of $D^2\varphi(x)$. (which is defined a.e on Ω.

Here we used Alexandrov thm:

$$\varphi(x)=\varphi(x_0)+\nabla\varphi(x_0)\cdot(x-x_0)+\frac{1}{2}(x-x_0)^T D^2\varphi(x_0)(x-x_0)+o(|x-x_0|^2)$$

for a.e x_0).

Let $M\subseteq\Omega$ be the subset s.t

φ is differentiable, $D_A^2\varphi$ is defined & invertible, & Lebesgue pt for $\det D_A^2\varphi$.

Then we have

(i). M is of full measure for $f(x)dx$, $\partial\varphi(M)$ is of full measure for $g(y)dy$

(ii) $\det D_A^2\varphi(x)=$ a.c part of $\det D_H^2\varphi$. & $\nabla\varphi_\#(\det D_A^2\varphi(x)dx)=1_{\partial\varphi(M)}dy$.

(iii) For a.e $x\in M$, $\det D_A^2\varphi(x)\cdot g(\nabla\varphi(x))=f(x)$.

We use this prop to prove Caffarelli's thm. i.e. why do we need convexity of the support of target measure?

Pf (of Caffarelli's thm). $\det D_H^2 \varphi = h(x)dx + \mu$. $\mu \perp m$.

use (ii) & (iii), $h(x) = \det D_A^2 \varphi(x) = \frac{f(x)}{g(\nabla\varphi(x))}$ a.e.

Just need to show $\mu = 0$. i.e. need to show $\forall N \subseteq \mathbb{R}^n$, with $m(N) = 0$

then $\det D_H^2 \varphi(N) = 0$. i.e $m(\partial\varphi(N)) = 0$.

First. $$\int_{\partial\varphi(N)} g(y)dy = \int_{\{x \in M : \nabla\varphi(x) \in \partial\varphi(N)\}} f(x)dx = \int_N f(x)dx = 0.$$

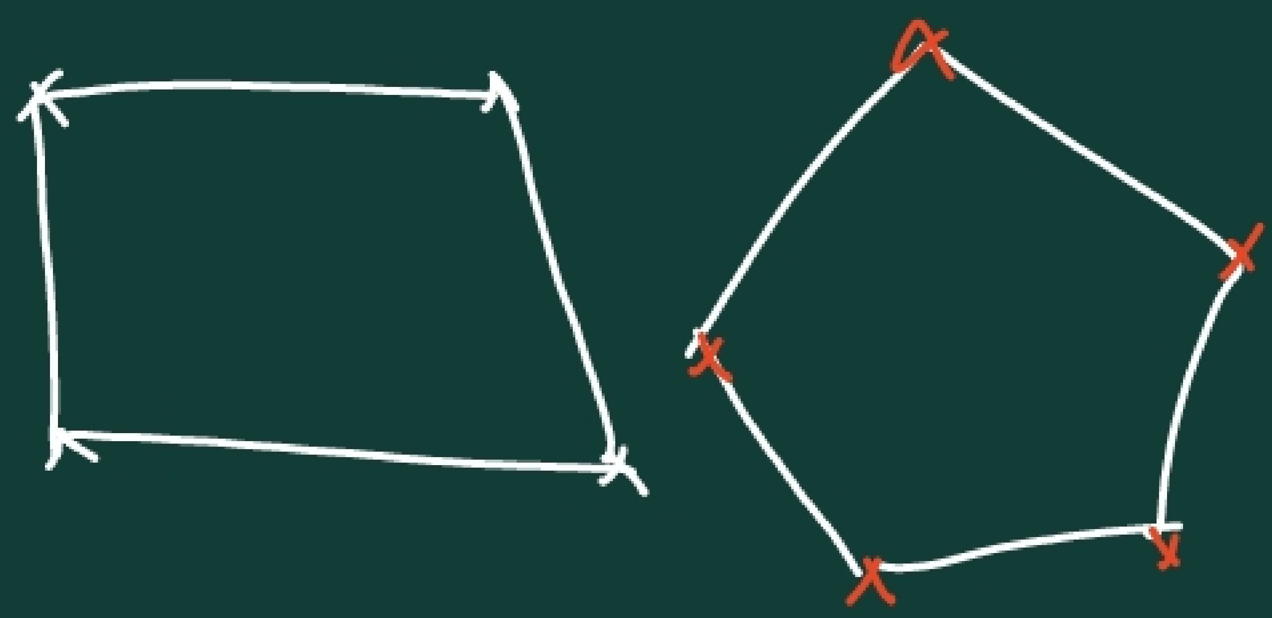
(we haven't used convexity of $\{g>0\}$ yet!)

Key pt: $\partial\varphi(N) \subseteq \{g>0\}$. Need the follwing fact about convex functions.

Lemma: (1) Let $x_0 \in \Omega$. Then for any extremal pt $p \in \partial\varphi(x_0)$, there exists a sequence $\{x_k\}_{k=1}^{\infty} \subseteq \Omega$. s.t. φ is differentiable at x_k, & $\nabla\varphi(x_k) \longrightarrow p$.

(2). Let $\Omega \subseteq \mathbb{R}^n$. be a closed bounded convex set. Then Ω is the convex hull of its extremal point.

Some explanations: We say $p \in \Omega$ is an extremal pt, if there doesn't exist $x_1, x_2 \in \Omega$, s.t. $p = \frac{x_1 + x_2}{2}$, unless $x_1 = x_2 = p$.



x : extremal pts.

: All the bdry is extremal pts.

We need to show $\partial\varphi(N) \subseteq \overline{\{g>0\}}$.

First. $\partial\varphi(M) \subseteq \overline{\{g>0\}}$. Let $x_0 \in N$. $p \in$ extremal pt of $\boxed{\partial\varphi\{x_0\}}$.

Also a closed bounded convex set.

$\exists\, x_k \in M$. s.t $\partial\varphi(x_k) \longrightarrow p$. i.e

$\forall\, x_0 \in N$, extremal pt of $\partial\varphi\{x_0\} \subseteq \overline{\{g>0\}}$.

Since $\partial\varphi(x_0)$ is generated from the extremal pt. convex

$\Rightarrow \partial\varphi(x_0) \subseteq \overline{\{g>0\}}$.

$$\int_{\partial\varphi(N)} g(y)dy = 0 \Rightarrow |\partial\varphi(N)| = 0$$

Since $g>0$ on $\partial\varphi(N)$.

proof of prop (sketch): (i) M is of full measure for $\mu = f(x)dx$.

$\underbrace{D\varphi \text{ exists}}_{a.e}$, $\underbrace{D_A^2\varphi}_{a.e}$ exists & $\underbrace{\text{invertible}}_{?}$. & $\underbrace{\text{Lebesgue pt.}}_{a.e}$

$D_A^2\varphi$ invertible $\Longleftrightarrow$ $D_A^2\varphi^*$ exists.

$$\mu(D_A^2\varphi \text{ not invertible}) = D\varphi^*_{\#}\nu(D_A^2\varphi \text{ not exist})$$
$$= \nu(D_A^2\varphi_* \text{ not exist}) = 0$$

since $D_A^2\varphi_*$ exist m-a.e. $\nu \ll m$.

$\det D_A^2\varphi(x)dx \overset{?}{=}$ a.c part of $\det D_H^2\varphi$.

To prove this, just need to show for a.e x_0 $\dfrac{\det D_H^2\varphi(B_r(x_0))}{m(B_r(x_0))} \longrightarrow \det D_A^2\varphi(x_0)$ a.e x_0. as $r \to 0$.

i.e for a.e x_0. $\dfrac{m(\partial\varphi(B_r(x_0)))}{m(B_r(x_0))} \longrightarrow \det D_A^2\varphi(x_0)$.

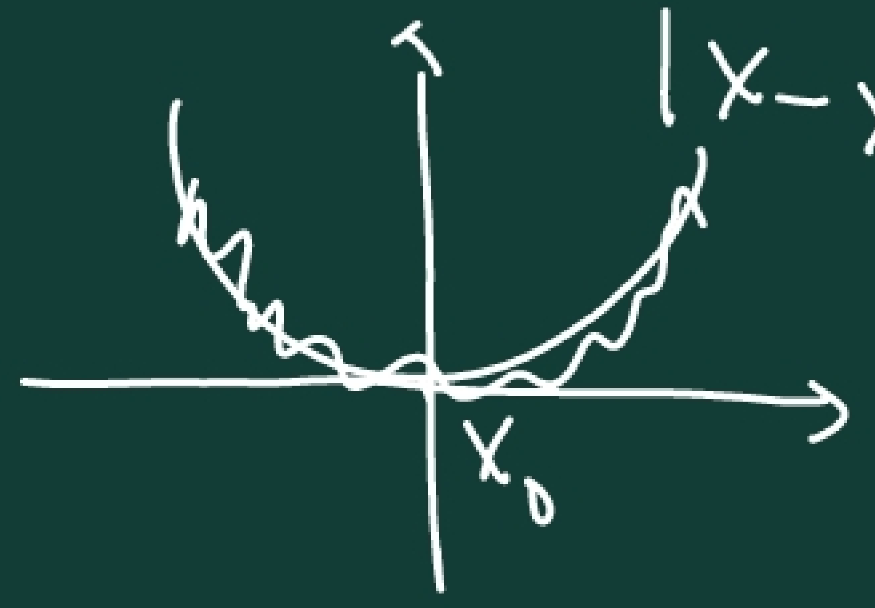
By Alexandrov thm, for a.e x_0, we can write down

$$\varphi(x)=\varphi(x_0)+\nabla\varphi(x_0)\cdot(x-x_0)+\frac{1}{2}(x-x_0)^T D^2\varphi(x_0)(x-x_0) + o(|x-x_0|^2)$$

WLOG, can assume $\varphi(x_0)=0$, $\nabla\varphi(x_0)=0$,

$$D^2\varphi(x_0)=I.$$

i.e $\varphi(x)=\frac{1}{2}|x-x_0|^2+o(|x-x_0|^2)$



$$B_{r(1-\epsilon)}(x_0)\subseteq \partial\varphi(B_r(x_0)) \subseteq B_{r(1+\epsilon)}(x_0)$$

i.e $\dfrac{m(\partial\varphi(B_r(x_0)))}{m(B_r(x_0))} \approx 1.$