Lemma (Hitchin)

If (M^4, w_1, w_2, w_3) $\begin{cases} w_1^2 = w_2^2 = w_3^2 \\ w_1 \wedge w_2 = 0 \\ w_1 \wedge w_3 = 0 \\ w_2 \wedge w_3 = 0 \end{cases}$ $\Updownarrow$

$w_i \in \Omega^2(M^4)$. symplectic, i.e. $w(X,Y)$ non-degenerate.

If $W_i \wedge W_j = 2\delta_{ij} V$, $V \in \Omega^4(M^4)$

and $dW_i = 0$

then $\exists (g, I, J, K)$ $\cancel{H}$K s.t.

$(W_1, w_2, w_3) = (W_I, w_J, W_K)$

or $(w_I, w_J, -w_K)$

Proof: Pointwisely,

$\mathrm{span}\{W_1, w_2, w_3\} \subset \Lambda^2_p(M)$

$\exists g$ s.t. $\{W_1, w_2, w_3\}$ SD.

i.e. $*_g w_i = w_i$.

Rescale g s.t.

$V = \mathrm{Vol}_g$.

$\Rightarrow$ Pointwisely, get g

$\Rightarrow$ recover I, J, K.

$I^2 = J^2 = K^2 = -\mathrm{Id}$, $IJ = K$.

If $IX = iX$, $IY = iY$

need to show that $I[X,Y] = i[X,Y]$

To prove this,

$$W_2([X,Y], Z) = W_3(I[X,Y], Z)$$

$$= W_2(L_X Y, Z)$$

$$= L_X(W_2(Y,Z))$$

$$- (L_X W_2)(Y, Z)$$

$$- W_2(Y, L_X Z)$$

$$= L_X(W_3(IY, Z))$$

$$- (L_X W_2)(Y, Z)$$

$$- W_3(IY, L_X Z)$$

$$= iL_X(W_3(Y,Z))$$

$$- iW_3(Y, L_X Z)$$

$$- [d(X \lrcorner W_1)](Y, Z)$$

$L_X W = X \lrcorner dW_1 - d(X \lrcorner W_1)$

$$= i(L_X W_3)(Y,Z)$$

$$+ iW_3(L_X Y, Z) - (d(X \lrcorner W_1))(Y, Z)$$

$$= \left[i\, \cancel{d(X \lrcorner \omega_3)} \right] (Y, Z)$$

$$- \cancel{d(X \lrcorner \omega_2)} (Y, Z)$$

$$+ i\, \omega_3 (L_X Y, Z)$$

$$X \lrcorner \omega_2 = \pm I^* (X \lrcorner \omega_3)$$

$$IX = iX \qquad = \pm i (X \lrcorner \omega_3)$$

$$= i\, \omega_3([X, Y], Z)$$

$$\Rightarrow I[X, Y] = i[X, Y].$$

$\Rightarrow$ I integrable (using $d\omega_2 = 0$, $d\omega_3 = 0$)

$+\ d\omega_1 = 0 \Rightarrow I$ kähler.

For same reason, get HK.

Now if $\omega_i \in \Omega^2(M)$ obtained by gluing. $d\omega_i = 0$.

$\omega_i \wedge \omega_j = a_{ij} V$, renormalize V s.t. $\det(a'_{ij}) = 1$.

Question: How to perturb ω_i s.t. we get HK.

Idea: For any $w_i \wedge w_j = a_{ij} V$

$|a_{ij} - \delta_{ij}| << 1$.

we can find a metric g s.t. w_i are self-dual and $\mathrm{Vol}_g = V$.

we can take

$\sqrt{(a_{ij})} = b_{ij}$ $w_i' = b_{ij} w_j$

keep Ω^+,

but make

$w_i' \wedge w_j' = a_{ij} V$

$w_i' = F(a_{ij})$

In general $dw_i = 0$

but $dw_i' \neq 0$

Now we consider

$d^+ \oplus d^*: \Omega^1 \to \Omega^+ \oplus \Omega^0$

Dirac operator

Cokernel $\mathrm{span}\{w_i\} \oplus \{1\}$

$\exists G: \Omega^+ \to \Omega^1 \oplus \mathrm{span}_{\mathbb{R}}\{w_i\}$

s.t.

$G_1 w_i' \in \Omega^1$

$G_2 w_i' \in \mathrm{span}_{\mathbb{R}}\{w_i\}$

$d^+(G_1 w_i') = w_i' + G_2 w_i'$

and $d^*(G_1 w_i') = 0$.

Then we want that to find $\eta^i \in \Omega^1(M)$ and $X^i \in \mathrm{span}_{\mathbb{R}}\{w_i\}$

s.t. $d(w_i + d\eta^i + X^i) = 0$

and $(w_i + d\eta^i + X^i) \wedge (w_j + d\eta^j + X^j) = 2\delta_{ij} V.$

$\iff (w_i + d^+\eta^i + X^i) \wedge \cdots$

$= 2\delta_{ij} V - (d^-\eta^i \wedge d^-\eta^j)$

$\iff w_i + d^+\eta^i + X^i$

$= F\left(\begin{smallmatrix}2\delta_{ij} \\ -d^-\end{smallmatrix}\eta^i \wedge d^-\eta^j\right)$

$F(a_{ij}) \simeq w_i$

If δ_{ij} is close to a_{ij}

then η^i, X^i are small

$\Rightarrow d^-\eta^i \wedge d^-\eta^j$ very small

By implicit function theorem,

if $|a_{ij} - \delta_{ij}| \ll 1 \Rightarrow$ we can find a solution.

Key point: If $d^*\eta^i = 0$

then $\|\eta^i\| \le \|d^+\eta^i\|$.

we also need that

$\omega_i \wedge \omega_j = a_{ij} V$

$\|a_{ij} - \delta_{ij}\| \ll 1$.

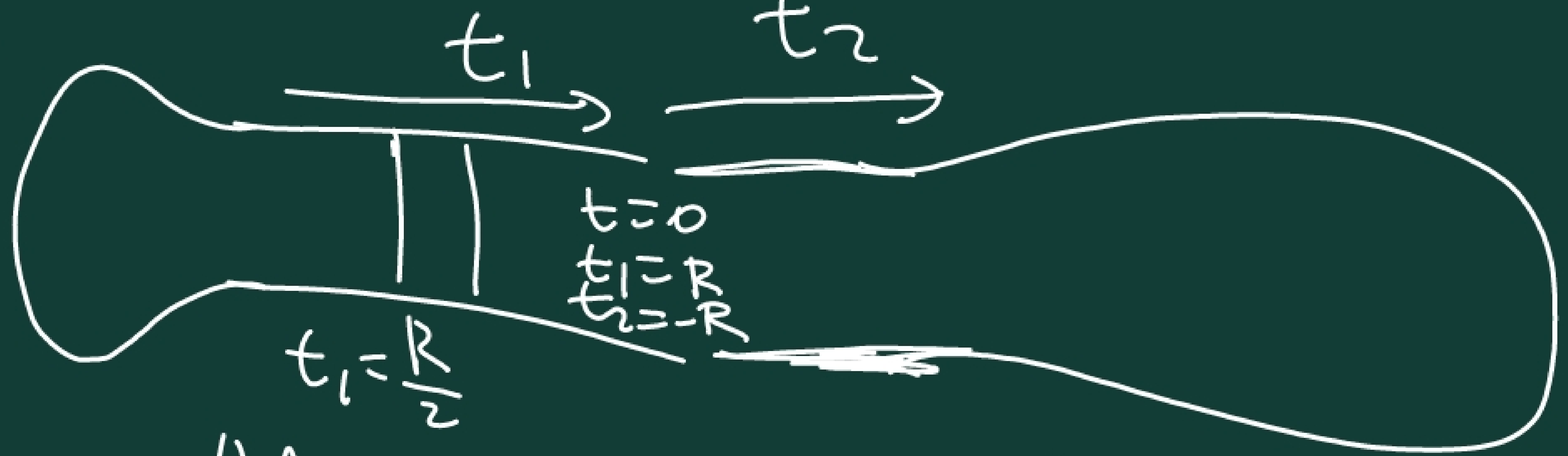
$\Delta = D^2$

$G_\Delta = \Delta^{-1}$

$G_D = D\, G_\Delta^{-1}$

Example (C.–Chen) (Based on an idea of Donaldson)

M ALH HK 4-manifold



$M_1 \quad = \quad M_2$

$t_1 \approx R \qquad t = t_1 - R = t_2 + R$.

$t_2 \approx -R$

$t \approx 0$ cut-off (ω_1^i), (ω_2^i)

Error $= O(e^{-\delta_0 R})$

$\Delta = (dd^* + d^*d) : \Omega^2 \to \Omega^2$

Choose $0 < \delta << \delta_0$

$\Delta_1 = e^{\delta t_1} \circ \Delta \circ e^{-\delta t_1} : W^{k,2}(M_1) \to W^{k-2,2}(M_1)$

$\Delta_2 = e^{\delta t_2} \circ \Delta \circ e^{-\delta t_2} : W^{k,2}(M_2) \to W^{k-2,2}(M_2)$

$\Delta = e^{\delta t} \circ \Delta \circ e^{-\delta t} : W^{k,2}(M) \to W^{k-2,2}(M)$

$\ker \Delta_1$ harmonic $\sim O(e^{\delta t_1}) \to$ must be $\mathrm{span}_{\mathbb{R}}\{w_i\}$

$\mathrm{coker}\, \Delta_1 = 0$

$\ker \Delta_2 = 0$

$\mathrm{coker}\, \Delta_2 = \mathrm{span}_{\mathbb{R}}\{w_i\}$.

$\mathrm{Ind}(W^{k,2}_{\delta})$

$- \mathrm{Ind}(W^{k,2}_{-\delta})$

$= \dim\{1, t\}$

$= 2$

$\mathrm{Ind}(W^{k,2}_{\delta})$

$= -\mathrm{Ind}(W^{k,2}_{-\delta})$

$\Rightarrow \mathrm{Ind}(W^{k,2}_{\delta}) = 1$

$\mathrm{coker} = 0$

$\Rightarrow \dim \ker = 1$

$\Rightarrow \ker = \mathrm{span}_{\mathbb{R}}\{w_i\}$

$\Delta_1 : (\ker)^{\perp} \to (\mathrm{coker})^{\perp}$

Invertible and

$\|\Delta_1^{-1}\| \leq C.$

Prop (Kovaler-Singer)

$$\Delta^* : \left(\chi\left(\tfrac{R}{2}\right)\ker\Delta_1 \oplus \chi\left(\tfrac{R}{2}\right)\ker\Delta_2 \right)^{\perp} \longrightarrow \left(\chi\left(\tfrac{R}{2}\right)\mathrm{coker}\,\Delta_1 \oplus \chi\left(\tfrac{R}{2}\right)\mathrm{coker} \right)^{\perp}$$

is also invertible. Moreover,

$\|\Delta\|^{-1} \leq C.$

Proof: It suffices to show that

$$\|u\|_{W^{k,2}} \leq c\|\Delta^* u\|_{W^{k-2,2}}$$

$\Rightarrow \Delta^*$ is injective.

However, we can dualize everything to show that the dual operator is injective

$\Rightarrow$ It's surjective.

Using contradiction argument.

If $\exists R_i \to \infty$

$\|u_i\|_{W_t^{k,2}} = 1, \Rightarrow \|u_i\|_{W_\delta^{k,2}} = 1.$

$\|\Delta^* u\|_{W_t^{k-2,2}} \to 0.$

Then $o(e^{-\delta t})$ $\underline{t_1 = R \quad t = 0 \quad t_2 = -R}$ $\leq o(e^{\delta t})$

$\forall A$

near $|t| \leq A$ $\quad u_i \to u_\infty$ in $L^2(A)$

$\Delta^* u_\infty = 0 \Rightarrow \Delta^* u_\infty$ in strong sense.

$\|u_\infty\|_{W_\delta^{k,2}} \leq 1. \Rightarrow u_\infty = 0.$

In particular

$$\|u_i\|_{W^{k,2}/|t|\le 1} \longrightarrow 0.$$

Now we cut off near $|t| \le 1$

$$u_i \approx u_{i,1} + u_{i,2}$$

supported in M_1 supported in M_2.

$$u_{i,1} \approx (\ker \Delta_1)^{\perp} \quad \delta^* (u_{i,1}) \approx (\operatorname{coker} \delta_1)^{\perp}$$

Use the estimate on M_1, M_2 individually

$$\Rightarrow \|u_i\| \le C\|\delta^* u_i\| \longrightarrow 0$$