$u: \mathbb{R}^3 \times S^1 \to \mathbb{R}$

If u is S^1-invariant

$$\|u\|_{W^{k,2}_\delta} \le C\left(\|\Delta u\|_{W^{k,2}_{\delta-2}} + \|u\|_{L^2(B_R)}\right)$$

If u is perpendicular to S^1-invariant

i.e. $\int_{\{x\}\times S^1} u = 0 \quad \forall x \in \mathbb{R}^3.$

Poincaré Lemma: If $\int_{S^1} u = 0$,

then $\int_{S^1} |u|^2 \le c \int_{S^1} |\nabla u|^2.$

Corollary. For product metric

If $\int_{S^1} u = 0$

and $u \in C_0^\infty(\mathbb{R}^3 \times S^1)$,

then $\forall \delta \in \mathbb{R}$, we have

$$\|u\|_{L^2_\delta} \le C\|\Delta u\|_{L^2_\delta}.$$

$$\left(\le C R^{-2} \|\Delta u\|_{L^2_{\delta-2}}\right.$$

$$\left.\text{If } u=0 \text{ on } B_R\right)$$

if $u = 0$ on B_R, $R \gg 1$.

Proof: $\mathbb{R}^3 = \bigcup_i \{2^i < r < 2^{i+2}\}$

Then for u supported on $\{2^i < r < 2^{i+2}\}$ we have $\int |u|^2 r^{-3-2\delta}$

$\leq CR^{-3-2\delta} \int |u|^2$

$\leq CR^{-3-2\delta} \int |\nabla u|^2$

$\leq CR^{-3-2\delta} \int u \cdot \nabla^2 u$

$\leq C \sqrt{\int |u|^2 r^{-3-2\delta}} \sqrt{\int |\nabla u|^2 r^{-3-2\delta}}$

$\Rightarrow \int |u|^2 r^{-3-2\delta} \leq \int |\nabla u|^2 r^{-3-2\delta}$

In general, χ_i cut-off function supported in $2^i < r < 2^{i+2}$, $|\nabla\chi_i| = O(r^{-1})$, $|\nabla^2\chi_i| = O(r^{-2})$

$\Rightarrow \int |u|^2 r^{-3-2\delta} \leq \sum_i \int |\chi_i u|^2 r^{-3-2\delta}$

$\leq C \sum_i \int |\nabla(\chi_i u)|^2 r^{-3-2\delta}$

$\leq C \sum_i \int \chi_i^2 |\nabla u|^2 r^{-3-2\delta}$

$+ C \sum_i \int |\nabla \chi_i|^2 \cdot |\nabla u|^2 r^{-3-2\delta}$

$+ C \sum_i \int |\nabla \chi_i|^2 |u|^2 r^{-3-2\delta}$

It's not hard to get the required estimate if $R \gg 1$.

In particular, without any condition, we have

$$\|u\|_{W^{k+2,2}_{\delta}} \le C\|\Delta u\|_{W^{k,2}_{\delta-2}} + \|u\|_{L^2(B_R)}$$

In particular, $\Delta : W^{k+2,2}_{\delta} \to W^{k,2}_{\delta-2}$ is Fredholm.

The next thing is

$\forall f \in W^{k,2}_{\delta-2}$, we need to find $u \in W^{k+2,2}_{\delta}$ s.t. $\Delta u = f \;\; \forall r > R$.

If ~~it's~~ invariant, same as before.

Now if $\int_{S^1} f = 0$.

In fact, when restricted to $\int_{S^1} = 0$

$\forall\ R' > R$

on $R' > r > R$,

we have $\int |u|^2 \leq c \int |Du|^2$.

In particular,

$\int |u|^2 + \int |Du|^2$

$\sim \int |Du|^2$

$W^{1,2}$

completion of C_0^∞ in any norm.

If we use $|Du|$ as norm,

then $\forall f$

$w \longrightarrow (w, f)_{L^2}$

is a bounded linear function.

$\Rightarrow \exists u$ s.t.

$\int (Du, Dw) = \int wf$

$\Rightarrow \Delta u = f$

This provides a Green operator G on $R<r<R'$.

Now assume that $f \in W^{k,2}_{\delta}(r>R)$ and $\int_{S^1} f = 0$, then $f\chi_{R'} \to f$ in $W^{k,2}_{\delta}(r>R)$

$$\|G(f\chi_{R'})\|_{W^{k+2,2}_{\delta}} \le c\|f\chi_{R'}\|_{W^{k,2}_{\delta}}$$

$$f\chi_{R'} \to f \text{ in } W^{k,2}_{\delta}$$

$$\Rightarrow G(f\chi_{R'}) \to u \text{ in } W^{k+2,2}_{\delta},$$

we define $u = Gf$

If $\Delta u = 0$ on $r > R \gg 1$.

and $\int_{S^1} u = 0$

we claim that

$u = O(e^{-\delta \cdot r})$

Proof: $\forall R'$, we choose $\chi_{R'}$ s.t $\begin{cases} \chi_{R'} = 0, \ \forall r < R' \\ \chi_{R'} = 1, \ \forall r > R'+1 \end{cases}$

$\int_{r > R'+2} |u|^2 r^{-3-2\delta}$

Then $\leq \int |\chi_{R'} u|^2 r^{-3-2\delta}$

$\leq C \int (\Delta |\chi_{R'} u|)^2 r^{-3-2\delta}$

$\leq C \int_{R' < r < R'+1} (|\nabla^2 u|^2 + |\nabla u|^2 + |u|^2) r^{-3-2\delta}$

$\leq C \int_{R'-1 < r < R'+2} |u|^2 r^{-3-\delta}$

$\Rightarrow \int_{r > R'+2} \leq C \left(\int_{R'-1} - \int_{R'+2} \right)$

$\Rightarrow (1+C) \int_{r > R'+2} \leq C \int_{R'-1}$

$\Rightarrow \int_{r > R'+2} \leq \left(\frac{C}{1+C} \right) \int_{R'-1}$

Applications

Thm (Minerbe)

Any hyperkähler

ALF-A_k 4-manifold
(asymptotic to mTN)
must be multi-Taub-NUT.

(Gibbons-Hawking)

$$\mathbb{R}^3 - \{p_1, \cdots, p_k\}$$

$$V = 1 + \sum_i \frac{\mu}{|x - p_i|}$$

$$S^1 \to E^4$$
$$\downarrow$$
$$\mathbb{R}^3 - \{p_i\}$$

$$g = V^{-1}\eta^2 + V(g_{\mathbb{R}^3})$$

Proof: On the mTN^E, x, y, z on $\mathbb{R}^3$ are harmonic. $g_M = g_E + O(r^{-\varepsilon})$

On M, we choose χ s.t. $\chi = 0$ on B_R, $\chi = 1$ for $r > 2R$.

$$\Delta(\chi x) = O(r^{-1-\varepsilon}) \in W^{k,2}_{-1-\varepsilon'} \quad \forall k \geq 1$$

$$\Delta : W^{k+2,2}_{1-\varepsilon'} \to W^{k,2}_{-1-\varepsilon'}$$

It's not hard to see that Δ on the dual space.

Harmonic function must decay.

$\Rightarrow$ Must be 0.

$\Rightarrow$ No cokernel in original space.

$\Rightarrow \exists\, u \in W^{k+2,2}_{1-\varepsilon'}$

s.t. $\Delta u = \Delta(X \cdot x)$

$\Rightarrow X \cdot x - u \approx x - W^{k+2,2}_{1-\varepsilon'}$

and $\Delta(X \cdot x - u) = 0$ Call this $\tilde{x}$.

for the same reason, we can find $\tilde{y}, \tilde{z}$ harmonic.

Now on E, $I: TM \to TM$ $I^*: T^*M \to T^*M$.

$I^* dx = \pm dy$, $J^* dx = \pm dz$, $K^* dx = \pm\, d\eta$

On M, Hodge Laplacian on 1-form = connection Laplacian

$\Rightarrow$ Commute with I^*.

$\Rightarrow$ $I^* d\tilde{x}$ harmonic 1-form.

$I^* d\tilde{x} - d\tilde{y}$ harmonic 1-form. $= O(r^{-\varepsilon'})$

Bochner technique $\Delta |I^* d\tilde{x} - d\tilde{y}|^2 \geq 0$

$\Rightarrow I^* d\tilde{x} = d\tilde{y}$.

For same reason

$J^* d\tilde{x} = d\tilde{z}$

$\vdots$

So $d\tilde{x}, d\tilde{y}, d\tilde{z}, K^* d\tilde{x}$

recover the v.n.

$\Rightarrow$ Recover all structures

Next case is the ALF-D_k case.

In this case

$$M \approx \overset{mTN}{\overset{\downarrow}{E}} / \mathbb{Z}_2$$

$$\begin{array}{c} S^1 \to E/\mathbb{Z}_2 \\ \downarrow \\ \mathbb{R}^3/\mathbb{Z}_2 \end{array}$$

Now x is not well-defined.

but x^2 is well defined.

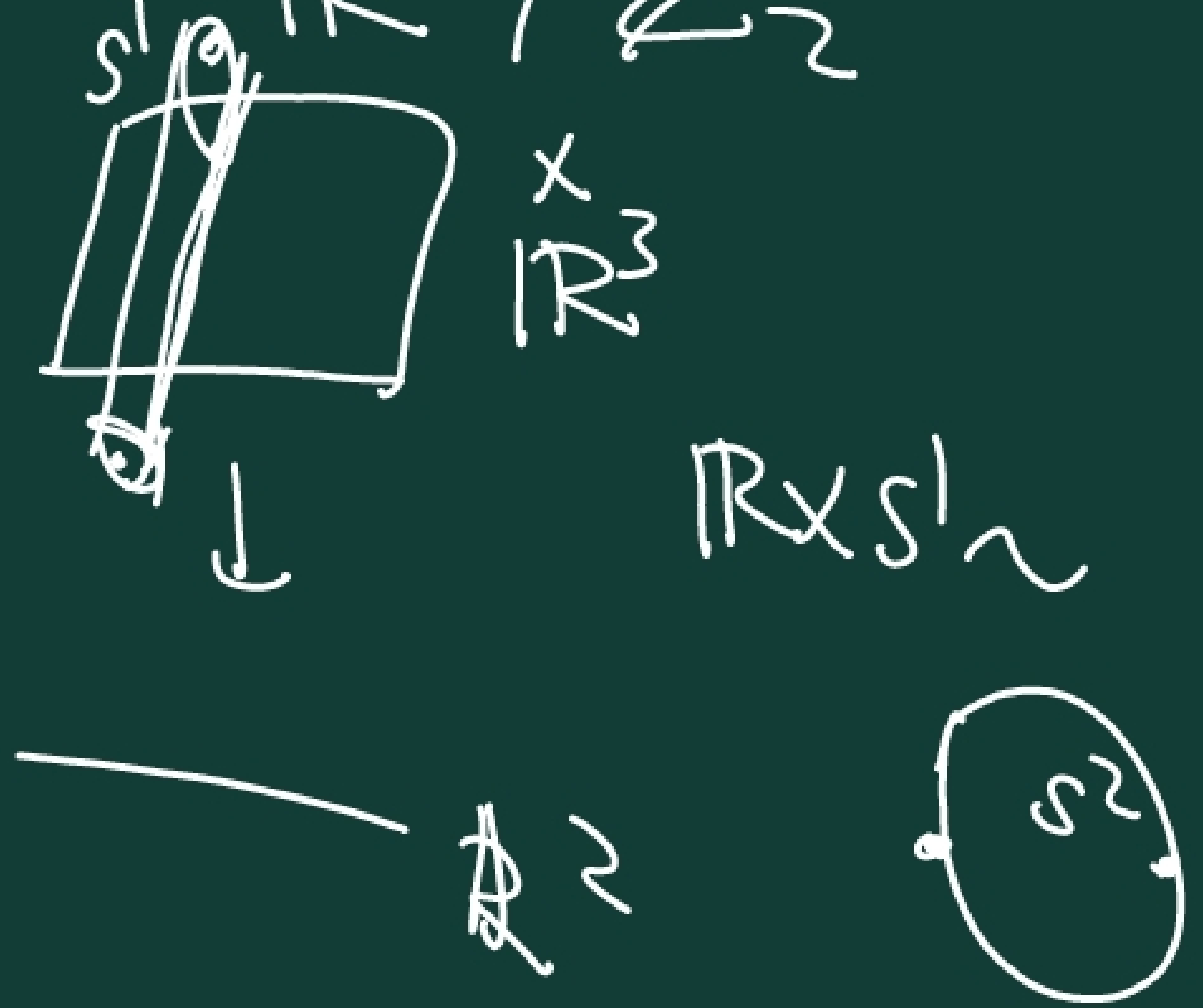
xy is well defined.

So x^2-y^2, x^2-z^2, xy, xz, yz. are well-defined harmonic function. Using a similar method, we can find harmonic functions on M asymptotic to these functions.

Then we can recover rational fiberation structure.

e.g.

$$\begin{array}{c} (\mathbb{R}^3 \times S^1)/\mathbb{Z}_2 \\ \downarrow \\ \mathbb{R}^2/\mathbb{Z}_2 \end{array}$$



using classification of rational fiberation (i.e. generic fiber is $\mathbb{CP}^1$),

we can understand the biholomorphic structure of its compactification.

we can do this for each $(aI+bJ+cK)$, $(a,b,c)\in S^2$

in a consistence way

So we recover the twistor space

$\Longrightarrow$ ALF-D_k must be standard.

M HK, Twistor space $M\times S^2$ for each $(a,b,c)\in S^2$, (M,a,b,c) has a complex sturcture $aI+bJ+cK$