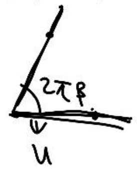
Def: $(M, g, I, J, K) \iff (g, I), (g, J), (g, K)$ kähler

$IJ = K$. $\omega_I, \omega_J, \omega_K$ kähler forms.

ALE model : $(\mathbb{R}^4/\Gamma, g, I, J, K)$ Γ finite subgroup of $SU(2)$

ALH model : $((0, +\infty) \times T^3, g, I, J, K)$ $v \in T^2 = \mathbb{C}/\Lambda$

ALG model:  $\times T^2 \Big/ \sim$ $(u, v) \sim (u \cdot e^{2\pi i \beta}, v \cdot e^{-2\pi i \beta})$

require $e^{-2\pi i \beta} \cdot \Lambda = \Lambda$

M is ALE $\iff \exists X = (\mathbb{R}^4/\Gamma, \underline{g}, \underline{I, J, K})$

A diffeomorphism $\Phi : X - B_R(0) \to M \setminus C$

$\| \nabla^k_{g_X} (\Phi^* g_M - g_X) \|_{g_X} = O(r^{-\varepsilon - k})$, $r(p) = d(0, p)$, $\varepsilon > 0$

$\| \nabla^k_{g_X} (\Phi^* I_M - I_X) \|_{g_X} = O(r^{-\varepsilon - k})$.

$\Phi^* J$ $\vdots$

$\Phi^* K$

Def: M is called a gravitational instanton $\iff$ M is hyperkähler

$\dim_{\mathbb{R}} M = 4$ M is non-compact

$\int_M |Rm|^2 < \infty$

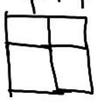
Thm (Sun - Zhang) Gravitational instantons are ALE, ALF, ALG, ALG*, ALH, ALH*

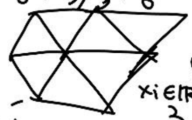
Kronheimer Minerbe Chen-Chen $|Rm| = O(r^{-2-\varepsilon})$

It can be proved that if $e^{-2\pi i\beta}\cdot\Lambda=\Lambda$, then (Λ,β) must be in the following list:

$\beta=\frac{1}{2}$ $\quad \frac{1}{4},\frac{3}{4}$ $\quad \frac{1}{6},\frac{1}{3},\frac{2}{3},\frac{5}{6}$

$\Lambda=\mathbb{Z}\oplus\mathbb{Z}\tau$





ALF $\quad$ ALG* $\quad$ ALH*

$x_i\in\mathbb{R}^3$ $\quad z\in$ $\quad t\in$

Gibbons-Hawking $\quad U=\mathbb{R}^3$, or $\mathbb{R}^2\times S^1$ or $\mathbb{R}\times T^2$

$V=1+\sum\frac{1}{|x-x_i|}$ $\quad a+b\log|z|$ $\quad a+bt$

V is a harmonic function on U

Find a S^1-fibration

$$U(1)=S^1\to X^4 \to U^3$$

Locally $X=S^1\times U_i$ (θ_i,x) $\quad X=S^1\times U_j$ (θ_j,x)

$\theta_i-\theta_j=f(x)$

Find a connection $\eta=d\theta_i+f_i(x)dx_i$

Curvature $d\eta\in\Omega^2(U^3)$ $\quad *_U d\eta=dV$ on $\mathbb{R}^3$

$g=V^{-1}\eta^2+V(dx^2+dy^2+dz^2)$ $\quad \omega_J=V dx\wedge dz+dy\wedge\eta \Rightarrow d\eta=\pm * dV \Rightarrow 0=\pm d*dV=\pm\Delta V$

$\omega_I=V dx\wedge dy+dz\wedge\eta$ $\quad \omega_K=V dy\wedge dz+dx\wedge\eta$.

Weighted Analysis on Cylinder.

N^n, $M^{n+1}=(0,+\infty)\times N^n$ ($t\in(0,+\infty)$) $\quad \Delta$ on M?

We start with function. $u(t,x)$ $\quad x\in N$

Q: $\Delta_M u=0$ on M^{n+1} $\quad u=\sum_i f_i(t)\cdot g_i(x)$

$\Delta_M=\left(\frac{\partial^2}{\partial t^2}+\Delta_N\right)$ $\quad 0=\Delta_M u(t,x)=\sum_i f_i''(t)\cdot g_i(x)+\sum_i f_i(t)\cdot\Delta_N g_i(x)$

Require: $\Delta_N g_i(x)=-\lambda_i^2 g_i(x)$

$f_i''(t)-\lambda_i^2 f_i(t)=0\Rightarrow f_i(t)=e^{\pm\lambda_i t}$

$$\exists\, u(t,x) = \sum_{\lambda_i \neq 0} A_i \underbrace{e^{\lambda_i t}} g_i(x) + B_i e^{-\lambda_i t} g_i(x) + (A_0 + B_0 t)\, g_0(x)$$

$\Delta_N g_i(x) = -\lambda_i^2 g_i(x)$, $\lambda_i \neq 0$ $\qquad \Delta g_0(x) = 0$, $g_0(x) = 1$

Q: $\Delta u = V$ $\quad \|u\|_{W^{2,2}} \leq \|V\|_{L^2}$?

λ_i are called "indicial roots" $\qquad u \approx e^{\delta t}$

~~Key estimate:~~ Def: $\|u\|_{L^2_\delta} = \int_M |u \cdot e^{-\delta t}|^2$

$\|u\|_{W^{k,2}_\delta} = \sum_{i=0}^{k} \int_M \|\nabla^i u \cdot e^{-\delta t}\|^2$

Key estimate: If $u \in C_0^\infty((0,+\infty)\times N)$, and $\underbrace{\delta \neq \pm\lambda_i}$, then $\exists C$ s.t. $\|u\|_{W^{2,2}_\delta} \leq C \|\Delta u\|_{L^2_\delta}$.

Firstly, $\|u\|_{W^{2,2}_\delta} \leq C(\|\Delta u\|_{L^2_\delta} + \|u\|_{L^2_\delta})$.

Proof: $(0,+\infty) = (0,2) \cup (1,3) \cup (2,4) \cup (3,5) \cup \cdots$

$$\|u\|_{W^{2,2}_\delta} \leq C \sum_i \left[\|u\|_{W^{2,2}((i-1,i+1)\times N)} e^{-\delta i} \right] \quad u \text{ on } (i-1,i+1)\times N$$

$t \in (i-1,i+1)$

$$\leq C \sum_i \left[\left(\|\Delta u\|_{L^2((i-2,i+2)\times N)} \right) + \|u\|_{L^2((i-2,i+2)\times N)} e^{-\delta i} \right] \qquad e^{\delta t} \approx e^{\delta i}$$

$$\leq C(\|\Delta u\|_{L^2_\delta} + \|u\|_{L^2_\delta})$$

Now we want to show that $\qquad u \in C_0^\infty$

$$\|u\|_{L^2_\delta} \leq C \|\Delta u\|_{L^2_\delta}$$

To prove this, $u \sim \sum u_i(t) \cdot g_i(x)$ $\qquad$ g_i orthornormal basis in $L^2(N)$

Fourier series.

$$\Delta u = \sum \left[u_i''(t) - \lambda_i^2 u_i(t) \right] \cdot g_i(x)$$

We only need to prove that $\exists C$ independent of i s.t. $\int_0^\infty |u_i(t)|^2 e^{-2\delta t} \leq C \int_0^\infty |u_i''(t) - \lambda_i^2 u_i(t)|^2 e^{-2\delta t}$

$$(u_i'' - \lambda_i^2 u_i)\, e^{-\lambda_i t}$$

$$= \left(e^{-2\lambda_i t}\left(u_i e^{\lambda_i t}\right)'\right)' \qquad \mu = \lambda_i, \quad \nu = -2\lambda_i$$

$$= \left[e^{\nu t}\left(u_i' e^{\mu t} + \mu u_i e^{\mu t}\right)\right]' \qquad 2\mu + \nu = 0$$

$$= \left[u_i' e^{(\mu+\nu)t} + \mu u_i e^{(\mu+\nu)t}\right]'$$

$$= u_i'' e^{(\mu+\nu)t} + (\mu+\nu) u_i' e^{(\mu+\nu)t} + \mu u_i' e^{(\mu+\nu)t} + \underset{-\mu^2}{\cancel{\mu(\mu+\nu)}}\, u_i e^{(\mu+\nu)t}$$