$u : (0,+\infty) \times N \to \mathbb{R}$

$$u = \sum_i f_i(t) \cdot g_i(x)$$

$$\Delta_N g_i(x) = -\lambda_i^2 g_i(x)$$

$$\Delta u = \sum_i \left(f_i''(t) - \lambda_i^2 f_i(t)\right) g_i(x)$$

If $u \in C_0^\infty((0,+\infty) \times N)$, then $\forall\, \delta \neq \pm\lambda_i$, we have

$$\|u\|_{W^{k,2}_\delta} \leq C \|\Delta u\|_{W^{k-2,2}_\delta} \quad \checkmark$$

It suffices to prove that

$$\|u\|_{L^2_\delta} \leq C \|\Delta u\|_{L^2_\delta}$$

$$\underline{f_i'' - \lambda_i^2 f_i}$$

$$= \left(\underline{e^{-2\lambda_i t}}\left(\underline{f_i e^{\lambda_i t}}\right)'\right)' e^{\lambda_i t}$$

Lemma: If $f \in C_0^\infty(\mathbb{R}(0,+\infty))$, $\forall \mu \neq 0$, we have

$$\int |f|^2 e^{\mu t} \leq C \int |f'|^2 e^{\mu t}.$$

Proof: $\left|\int_0^\infty f^2 e^{\mu t}\right| = \int_0^\infty f^2 e^{\mu t}$

$$= \left|\frac{1}{\mu}\int_0^\infty f^2 (e^{\mu t})'\right|$$

$$= \left|\frac{1}{\mu} f^2 e^{\mu t}\Big|_0^\infty - \frac{1}{\mu}\int_0^\infty (f^2)' e^{\mu t}\right|$$

$$= \frac{2}{\mu}\left|\int_0^\infty f\cdot f'\cdot e^{\mu t}\right|$$

$$\leq \frac{2}{\mu}\sqrt{\left(\int_0^\infty f^2 e^{\mu t}\right)}\sqrt{\left(\int_0^\infty (f')^2 e^{\mu t}\right)}$$

$$\Rightarrow \int_0^\infty f^2 e^{\mu t} \leq \frac{4}{\mu^2}\int_0^\infty (f')^2 e^{\mu t}$$

$$\Rightarrow \int_0^\infty f_i^2 e^{-2\delta t}$$

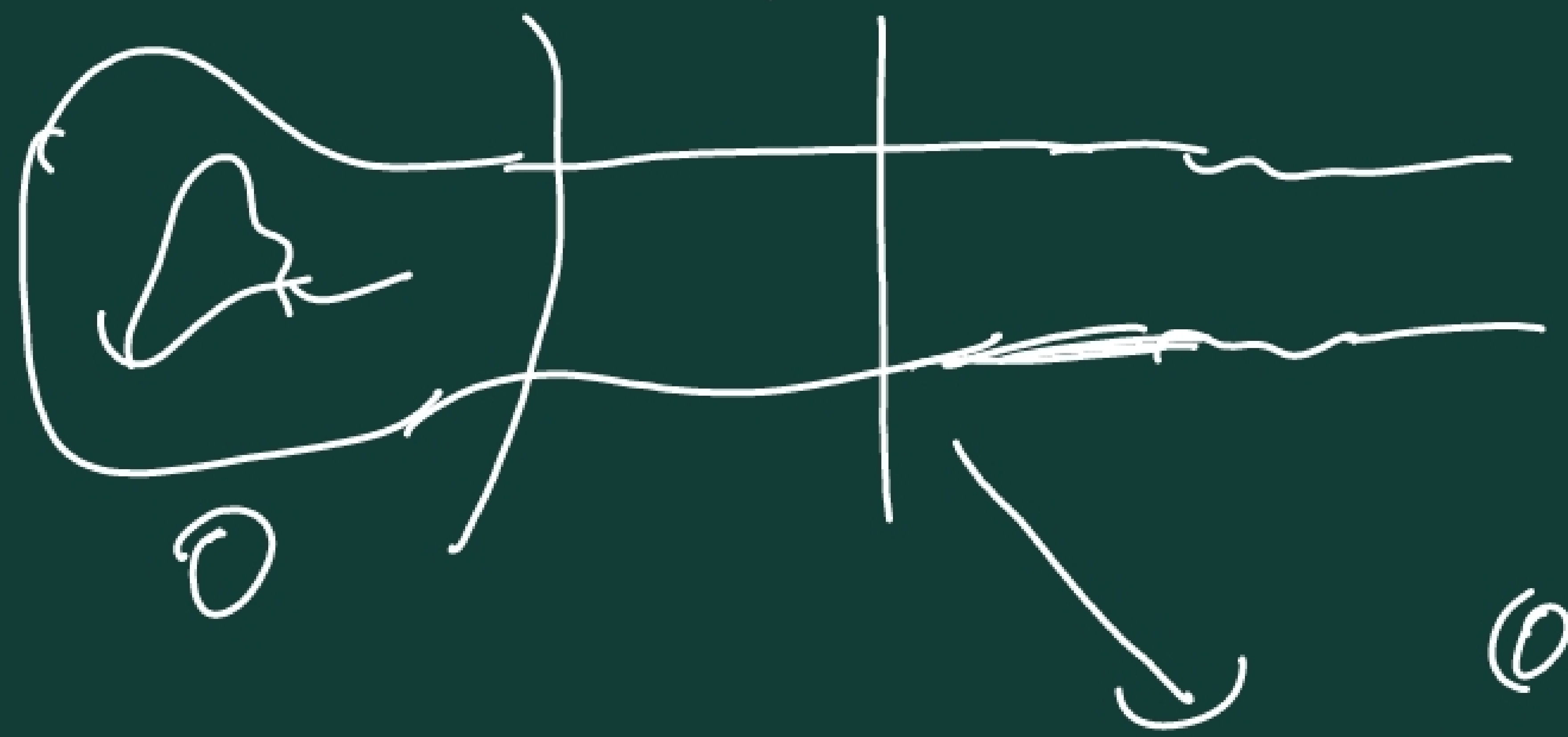
$$= \int_0^\infty (f_i e^{\lambda_i t})^2 e^{-2\delta t - 2\lambda_i t}$$

$$\leq \frac{4}{(2\delta + 2\lambda_i)^2} \int_0^\infty \left[(f_i e^{\lambda_i t})'\right]^2 e^{-2\delta t - 2\lambda_i t}$$

$$= \frac{4}{(2\delta + 2\lambda_i)^2} \int_0^\infty \left[(f_i e^{\lambda_i t})' e^{-2\lambda_i t}\right]^2 e^{-2\delta t + 2\lambda_i t}$$

$$\leq \frac{4}{(2\delta + 2\lambda_i)^2} \cdot \frac{4}{(2\delta - 2\lambda_i)^2} \int_0^\infty \left[\left((f_i e^{\lambda_i t})' e^{-2\lambda_i t}\right)' e^{\lambda_i t}\right]^2 \cdot e^{-2\delta t}$$

Now assume $M =$

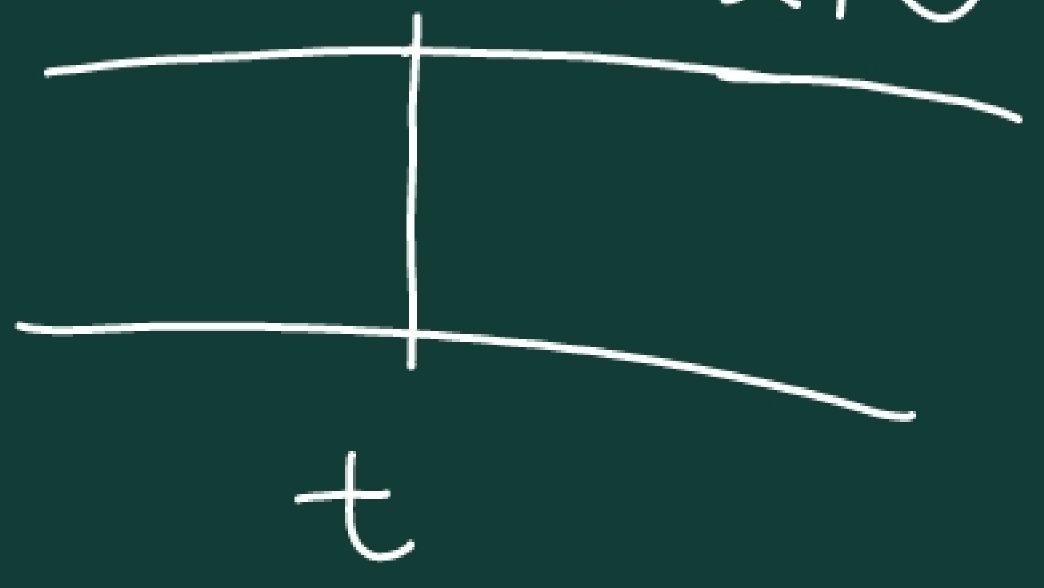


$g_M = g_{(0,+\infty)\times N} + O(e^{-\delta_0 t})$

$\delta_0 > 0.$

$M - O \cong (0,+\infty) \times N$

$(0,+\infty) \times N$



Define τ (weight function)

smooth, $\tau = t$ for all $t > T$.

Now we define $\|u\|_{L^2_\delta(M)} = \int |u|^2 e^{-2\delta\tau}$

$$\|u\|_{W^{k,2}_\delta(M)} = \sum_{i=0}^{k} \|D^i u\|_{L^2_\delta(M)}.$$

Key estimate on M

$\exists ! C > 0$ s.t. $\forall u \in C_0^\infty(M)$, (using the denseness of C_0^∞ in $W^{k,2}_\delta(M)$ this holds for all $u \in W^{k,2}_\delta(M)$)

$$\|u\|_{W^{k,2}_\delta(M)} \leq C\left[\|u\|_{L^2(\tau < T)} + \|\Delta_M u\|_{W^{k-2,2}_\delta(M)}\right] \checkmark$$

Proof: Choose χ cut-off $\chi(\tau - T) = 1$, $\forall \tau > T+1$
$\chi(\tau - T) = 0$, $\forall \tau < T$.

For T sufficient large (TBD),

$$\|\chi(\tau-T)u\|_{W^{k,2}_\delta(M)} \leq C\|\Delta_{[0,t_0)\times N}[\chi(\tau-T)u]\|_{W^{k-2,2}_\delta(M)} \sim g^{ij}D_{ij}$$

$$\leq C\|\Delta_M(\chi(\tau-T)u)\|_{W^{k-2,2}_\delta} + Ce^{-\delta_0 T}\|\chi(\tau-T)u\|_{W^{k,2}_\delta(M)}$$

Choose $T \gg 1$ s.t $Ce^{-\delta_0 T} < \frac{1}{2}$

$$\Rightarrow \| X(\tau - T) u \|_{W^{k,2}_\delta} \le C \| \Delta (X(\tau-T) u) \|_{W^{k-2,2}_\delta}$$

$\tau > T+1$

$$\le C\|\Delta u\|_{W^{k-2,2}_\delta} + C\|\Delta(X(\tau-T)u)\|_{W^{k-2,2}_\delta}(\tau<T+1)$$

$$\le C\|\Delta u\|_{W^{k-2,2}_\delta} + C\|\Delta(X(\tau-T)u)\|_{W^{k-2,2}}(\tau<T+1)$$

$$\le C\|\Delta u\|_{W^{k-2,2}_\delta} + C\|u\|_{W^{k,2}}(\tau<T+1)$$

$$\Rightarrow \|u\|_{W^{k,2}_\delta} \le C\|\Delta u\|_{W^{k-2,2}_\delta} + C\|u\|_{W^{k,2}}(\tau<T+1)$$

$$\le C\|\Delta u\|_{W^{k-2,2}_\delta} + C\|u\|_{L^2}(\tau<T+2) + C\|\Delta u\|_{W^{k-2,2}}(\tau<T+2)$$

We now show that $\Delta_M : W^{k,2}_\delta \to W^{k-2,2}_\delta$ is Fredholm.

To prove that $\dim \ker < +\infty$, it suffices to show that the unit ball is compact.

$\forall\, u_i \in \ker$ s.t. $\|u_i\|_{W^{k,2}_\delta} = 1$.

Rellich Lemma: $W^{k,2}_\delta(M) \to W^{k,2}(\tau < 2T) \hookrightarrow L^2(\tau < T)$ is compact.

$\exists$ a subsequence $u_i' \to u_\infty$ in $L^2(\tau < T)$.

$\Rightarrow \|u_i' - u_j'\|_{W^{k,2}_\delta} \leq C\|\Delta_M(u_i' - u_j')\|^{\to 0} + C\|u_i' - u_j'\|_{L^2(\tau<T)} \to 0$

Now we consider V, the perpendicular space of $\ker \Delta$ in $W^{k,2}_\delta$.

Claim: $\exists C > 0$ s.t. $\forall u \in V$

$$\|u\|_{W^{k,2}_\delta} \le C \|\Delta u\|_{W^{k-2,2}_\delta} \checkmark$$

Proof: Suppose this is not true, then $\exists u_i$ s.t.

$$\|u_i\|_{W^{k,2}_\delta} = 1 \quad \text{and} \quad \|\Delta u_i\|_{W^{k-2,2}_\delta} \to 0$$

$\exists$ subsequence, $u_i' \to u_\infty$ in $L^2(\tau < T)$

Then $\|u_i' - u_j'\|_{W^{k,2}_\delta} \leq C\|\partial u_i' - \partial u_j'\|_{W^{k-2,2}_\delta} + C\|u_i' - u_j'\|_{L^2(\tau<T)} \to 0$

So $u_i' \to u_\infty'$ in $W^{k,2}_\delta$. $\Rightarrow \partial u_i' \to \partial u_\infty'$, $\partial u_i' \to 0$

$\Rightarrow \partial u_\infty' = 0 \Rightarrow u_\infty' \in \ker$.

$u_\infty' \in V \Rightarrow u_\infty' = 0$. Moreover $\|u_\infty'\| = 1$. Contradiction.

Now we show that ∂ has a closed range ✓

$\forall\, v_i = \partial u_i$, and $v_i \to v_\infty$ in $W^{k-2,2}_\delta$.

$\exists\, \tilde{u}_i = u_i + w_i$, $w_i \in \ker \partial$, s.t. $\tilde{u}_i \in V$.

Then $\|\tilde{u}_i - \tilde{u}_j\| \leq C\|\partial\tilde{u}_i - \partial\tilde{u}_j\| \to 0$

$\Rightarrow \tilde{u}_i \to u_\infty$

$\Rightarrow \partial\tilde{u}_i \to \partial u_\infty$, $\partial\tilde{u}_i \to v_i$

$\Rightarrow v_i = \partial u_\infty$

Now we can actually characterize the range.

We consider the case $\Delta : W^{2,2}_\delta \to L^2_\delta$ first.

The range is closed. We need to understand the perpendicular space $\mathcal{U}$ in L^2_δ.

$$u \in \mathcal{U} \quad \text{iff} \quad (u, \Delta v)_{L^2_\delta} = 0 = \int u \cdot (\Delta v) \cdot e^{-2\delta t}$$

$u \in L^2_\delta$

$$= \int (u\, e^{-2\delta t}) \cdot \Delta v \iff \Delta(u \cdot e^{-2\delta t}) = 0 \text{ in the distribution sense}$$

$\iff$ u is smooth and $\Delta(u \cdot e^{-2\delta t})$ in the usual sense.

Thus f is in the range

iff $\int f\, u\, e^{-2\delta t} = 0$

$\forall u \in L^2_\delta$ s.t. $\Delta(u e^{-2\delta t}) = 0$

$\Updownarrow$

$u \cdot e^{-2\delta t} \in L^2_{-\delta}$ s.t. $\Delta(u \cdot e^{-2\delta t}) = 0$

In conclusion, $f \in L^2_\delta$ is in the range of $\Delta : W^{2,2}_\delta \to L^2_\delta$

$\Longleftrightarrow$ $\forall w \in L^2_{-\delta}$ s.t. $\Delta w = 0$

we have $\int_M f w = 0$

Thus, the range has finite codim, i.e.

$\mathrm{Coker} = L^2_\delta / \mathrm{Im}\,\Delta$ has finite dim. The dimension

$= \dim \ker(\Delta : W^{2,2}_{-\delta} \to L^2_{-\delta})$

In fact $\forall k>0$, f is in the image

$$\Delta : W^{k,2}_{\delta} \longrightarrow W^{k-2,2}_{\delta}$$

$l>0$

iff $\forall\, w \in W^{l,2}_{-\delta}$ s.t. $\Delta w = 0$,

we have $\int f w = 0$.

To see this, we restrict f to L^2_δ, then we find $\Delta u = f$. Then we improve regularity of u by

$$\|u\|_{W^{k,2}_{\delta}} \le C\|u\|_{L^2_\delta} + \|\Delta u\|_{W^{k-2,2}_{\delta}}.$$

Next time:

Figure out what happens if δ changes.