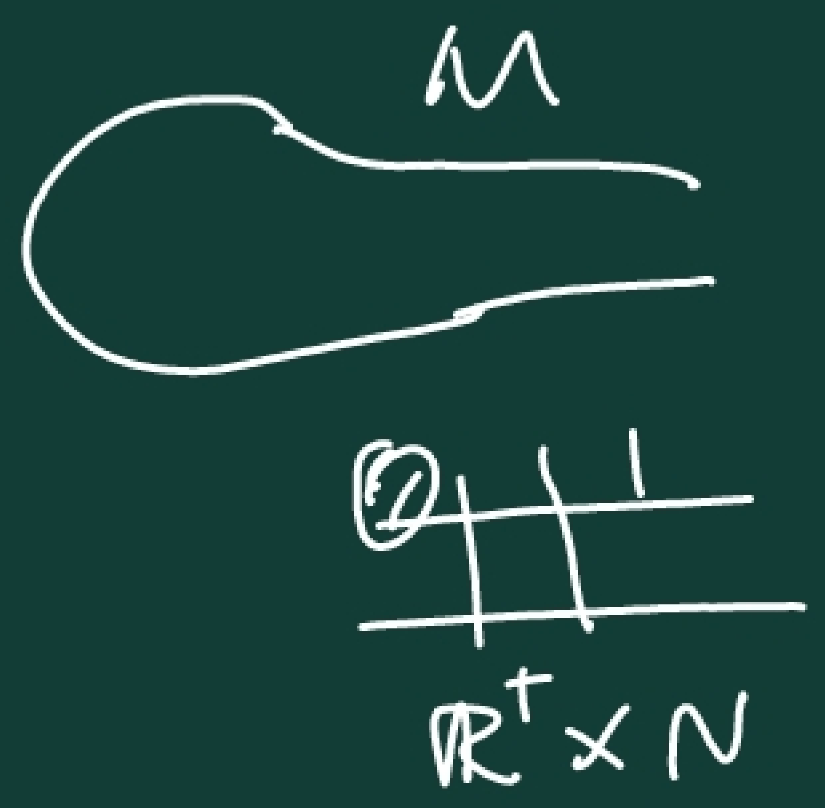

$$g_M = g_{\mathbb{R}^+\times N} + \boxed{O(e^{-\delta_0 t})}$$

$$\Delta_M : W^{k+2,2}_{\delta} \to W^{k,2}_{\delta} \quad \text{Fredholm. } \forall \delta \neq \pm\lambda_i$$

$$\varphi \in \mathrm{Im}\,\Delta_M \Leftrightarrow \int \varphi\,\psi = 0 \quad \forall\ \psi \in L^{k+2,2}_{-\delta} \text{ s.t. } \Delta\psi = 0$$

Thm: Suppose $\delta_1 < \delta_2$ are two weights with $\delta_i \neq \pm\lambda_j$,
$\delta_2 - \delta_1 < \delta_0$, assume that $(\delta_1, \delta_2) \cap \{\pm\lambda_i\} = \{\lambda'_i\}$
$\Delta g'_i = \lambda_i'^2 g'_i$, $\exists \chi$ s.t. $\begin{cases} \chi = 0, \forall t < T \\ \chi = 1, \forall t > T+1 \end{cases}$ $T \gg 1$.

Then

$$\begin{array}{ccc}
W^{k+2,2}_{\delta_1} \oplus \chi e^{\lambda_i' t} g_i' & \xrightarrow{\Delta_M} & W^{k,2}_{\delta_1} \\
\downarrow & & \downarrow \\
W^{k+2,2}_{\delta_2} & \xrightarrow{\Delta_M} & W^{k,2}_{\delta_2}
\end{array}$$

$O(e^{\delta_1 t})$

$O(e^{\lambda_i' t})$

$D^2 \to O(e^{\lambda_i' t}) \to W^{k,2}_{\lambda_i'-\delta_0}$

$\delta_1 < \delta_2$

on M.

Claim: $\ker \Delta_M \downarrow\wr \ker \Delta_M$ $\quad$ $\operatorname{coker} \Delta_M \downarrow\wr \operatorname{coker} \Delta_M$

Cor If $u \in W^{k+2,2}_{\delta_2}$ but $\Delta_M u \in W^{k,2}_{\delta_1}$, then $u \in W^{k+2,2}_{\delta_1} \oplus (\chi e^{\lambda_i' t} g_i)$

Corollary: $\operatorname{Ind}(W^{k+2,2}_{\delta_2} \to W^{k,2}_{\delta_2}) = \operatorname{Ind}(W^{k+2,2}_{\delta_1} \to W^{k,2}_{\delta_1}) \pm \dim(e^{\lambda_i t} g_i)$

Prop: $\forall \delta \neq \pm\lambda_i$, $\exists T >> 1$ s.t.

$$\exists G: W^{k,2}_\delta(t>T) \longrightarrow W^{k+2,2}_\delta(t>T)$$

s.t. $\Delta_{\mathbb{R}^+\times N} G_{\mathbb{R}^+\times N} u = u$, and $\|Gu\|_{W^{k+2,2}_\delta(t>T)} \le C\|u\|_{W^{k,2}_\delta}$

Lemma: $\forall \delta \neq 0$, $\exists S: L^2_\delta((0,+\infty)) \longrightarrow L^2_\delta((0,+\infty))$

s.t. $(Su)' = u$ and $\|Su\|_{L^2_\delta} \le C\|u\|_{L^2_\delta}$

Proof: If $\delta > 0$, then we define $(Su)(t) = \int_0^t u(\tau)\,d\tau$.

First, we assume $u \in C_0^\infty \xrightarrow{} L^2_\delta$, then $Su = 0$ near $t=0$, and $Su = \text{const.}$ near ∞, so $\int |Su|^2 e^{-2\delta t} < +\infty$.

we use C_0^∞ to app Su.

then $\int |Su|^2 e^{-2\delta t} \leq C \int |u|^2 e^{-2\delta t}$.

Next we consider $\delta < 0$, then

we define $Su(t) = -\int_t^\infty u(\tau)\, d\tau$.

We also assume that $u \in C_0^\infty$

$Su(t) = 0$, $t \gg 1$, but $Su(t) = \text{const}$ near $t = 0$

$$\int_0^\infty (Su)^2 e^{-2\delta t} = \frac{1}{-2\delta} \int_0^\infty (Su)^2 (e^{-2\delta t})' = \frac{1}{-2\delta} [\underbrace{(Su)^2 \cdot e^{-2\delta t}]\Big|_0^\infty}_{\leq 0}$$

$$- \frac{1}{-2\delta} \int_0^\infty 2(Su) \cdot u e^{-2\delta t} \leq \frac{1}{|\delta|} \left(\int_0^\infty (Su)^2 e^{-2\delta t} \right)^{1/2} \left(\int_0^\infty u^2 e^{-2\delta t} \right)^{1/2}.$$

Recall that

$$u = \sum f_i(t) g_i(x)$$

$$\Box u = \sum (f_i'' - \lambda_i^2 f_i) g_i(x)$$

$$= (((f_i e^{-\lambda_i t})' e^{2\lambda_i t})' e^{-\lambda_i t}) g_i(x).$$

So we integrate twice to get G.

Then previous G was $G_{\mathbb{R}^+\times N}$,

in fact we have G_M.

To see this

$$\Delta_M u = \Delta_{\mathbb{R}^+\times N} u + O(e^{-\delta t})\cdot D^2 u$$
$$+ O(e^{-\delta t})\cdot Du + O(e^{-\delta t})\cdot u.$$

$G_{\mathbb{R}^+\times N}$ is the converse of $\Delta_{\mathbb{R}^+\times N}$.

The error term $\leq C e^{-\delta T} \|u\|_{W^{2,2}_\delta}$.

Inverse function theorem $\Rightarrow \forall T \gg 1$, Δ_M is invertible.

$G_M = \Delta_M^{-1}$.

To show surjectivity of $\ker \Delta_M \to \ker \Delta_M$, we assume

$\Delta_M u = 0$, $u \in W^{k,2}_{\delta_2}$.

Then $\Delta_{\mathbb{R}^+ \times N} u \in W^{k,2}_{\delta_2 - \delta_0} \subset W^{k,2}_{\delta_1}$

$\exists\, w \in W^{k,2}_{\delta_1} (t > T)$ s.t.

$\Delta_{\mathbb{R}^+ \times N} w = \Delta_{\mathbb{R}^+ \times N} u$.

Then $\Delta_{\mathbb{R}^+ \times N}(w - u) = 0$

and $w - u \in W^{k+2,2}_{\delta_2}$.

$w - u = \sum f_i\, g_i$

$\Rightarrow f_i'' - \lambda_i^2 f_i = 0$

$\Rightarrow f_i = e^{\pm \lambda_i t}$

$\Rightarrow w - u = \sum e^{\pm \lambda_i t} g_i$

If $\pm \lambda_i > \delta_2$, we throw away it because $e^{\pm \lambda_i t} g_i \notin W^{k+2,2}_{\delta_2}$

If $t\lambda_i < \delta_2$, we write it as $\begin{cases} \lambda_i', & \delta_1 < t\lambda_i < \delta_2 \\ \text{error term.}, & t\lambda_i < \delta_1 \end{cases}$

$\Rightarrow w - u = \sum c_i e^{\lambda_i' t} g_i' + \text{error term.}$

$\|\text{error term}\|_{L^2_{\delta_1}(T' < t < T'+1)} \quad T' > T.$

$\leq C \|\text{error term}\|_{L^2_{\delta_1}(T'+1 < t < T'+2)}$

for $C < 1$.

$\Rightarrow \text{error term} \in W^{k+2,2}_{\delta_1}$.

$\Rightarrow u = \sum (c_i' e^{\lambda_i t} g_i') X + W^{k+2,2}_{\delta_1}$

Next, we show surjecivity of

$\operatorname{Coker} \partial_M \to \operatorname{Coker} \partial_M$.

$\forall u \in W^{k,2}_{\delta_2}$.

$(Xu) \in W^{k,2}_{\delta_2}(t > T)$.

Then $\exists\, w \in W^{k,2}_{\delta_2}(t>T)$ s.t.

$\Delta_M w = u$ on $t>T$

Then $\Delta_M(Xw) - u$

is compactly supported.

$\Rightarrow \ \in W^{k,2}_{\delta_1}$

$\Rightarrow u = \mathrm{Im}\,\Delta_M(Xw) + \big(u - \Delta_M(Xw)\big)$

In this quotient $[u] = [u - \Delta_M(Xw)]$

Then we show the injectivity of coker.

If $u \in W^{k,2}_{\delta_1}$,

$\exists\, w \in W^{k,2}_{\delta_2}$

s.t. $u = \Delta_M w$.

We solve $V \in W^{k,2}_{\delta_1}(t>T)$

s.t $\Delta_M V = u$ for $t>T$

$\Rightarrow \Delta_M(w - V) \overset{t>T}{=} 0$ and $w - V \in W^{k,2}_{\delta_2}$

$\Rightarrow w - V \in W^{k+2,2}_{\delta_1} \oplus X\,\mathrm{edit}^{\delta_2}_{\delta_1}$

Then $u = \Delta_M w$

$= \Delta_M (\underbrace{w - X}V) \in \operatorname{Im} \Delta_M$

$+ \Delta_M (XV) \in \operatorname{Im} \Delta_M$

$\Rightarrow [u] = 0$. This proves the injectivity.

We have worked out theories on asyptotically cylindrical manifolds. In fact, this is true for all gravitational instantons.

e.g. ALE

2 ALG

3 ALF

$\downarrow$

$\|u\|^2_{L^2_\delta} = \int |u|^2 r^{-4-2\delta}$

$u \sim r^\delta \quad \nabla u \sim r^{\delta-1}$

$W^{k,2}_{\delta} = \sum \|\nabla^i u\|_{L^2_{\delta-i}}$

Then $W^{k+2,2}_{\delta} \xrightarrow{\Delta_M} W^{k,2}_{\delta-2}$

Fredholm and

$$\begin{array}{ccc} W_{\delta_1} \oplus \chi r^{\lambda_i} g_i & \xrightarrow{\Delta_M} & W_{\delta_1} \\ \downarrow & & \downarrow \\ W_{\delta_2} & \xrightarrow{\Delta_M} & W_{\delta_2} \end{array}$$

ker, coker $\cong$

ALG:

$($  $\times T^2)/\sim$

Two pieces: T^2 invariant.

$\underbrace{\int_{T^2} u = 0}$ Always very good. reduce to $\mathbb{R}^2$

harmonic $\Rightarrow$ exponential decay.

Poincare lemma If $\int_{T^2} u = 0$, then $\int_{T^2} |u|^2 \le C \int_{T^2} |\nabla u|^2$ (Next time, use this to get estimates)