# LECTURE I. INTRODUCTION

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## Contents



## 1. Motivations

This is the first of a series of lectures notes for my course at USTC. In this series of notes, I will try to explain the idea and main techniques in the non-Archimedean pluripotential theory à la Boucksom–Jonsson to mathematicians working on complex geometry. Therefore, we have to assume some familiarity with complex manifolds and the usual pluripotential theory. The excellent book [GZ17] is sufficient for most parts of these notes.

In complex geometry, people study complex manifolds, which are *concrete* objects in the sense that one can imagine them: just glue unit balls in $\mathbb{C}^n$ together. Put things more abstractly, one can begin the study of complex geometry from the valued field $(\mathbb{C}, |\bullet|)$, where $|\bullet|: \mathbb{C} \to [0,\infty)$ is the usual absolute value. There is a canonical way to generate the unit balls in $\mathbb{C}^n$ from the data $(\mathbb{C}, |\bullet|)$ and hence generate all complex manifolds by gluing. In other words, complex geometry is a geometry relative to $(\mathbb{C}, |\bullet|)$.

In these notes, I will talk about some less concrete and less familiar objects: spaces over a non-Archimedean valued field $(k, |\bullet|)$. As three examples to keep in mind, one could take $k = \mathbb{Q}_p$ and $|\bullet|$ as the $p$-adic valuation; or one could take $k = \mathbb{C}((t))$ and $|\bullet|$ as the $t$-adic valuation; or one could take $k$ to be any field while $|\bullet|$ is the trivial valuation. In these cases, one could similarly develop a full theory of geometry relative to these data. But why bother?

This question can be answered from different angles. I will talk about a few of my favorite motivations.

First of all, a motivation from geometric representation theory. In the case of $\mathbb{Q}_p$, there is a famous object called *Drinfeld's upper half plane* $\Omega$ of dimension $d$, which can be constructed using non-Archimedean geometry. It can be regarded as an analytic space relative to $\mathbb{Q}_p$. The general framework established by Berkovich allows us to talk about étale cohomologies of the local systems $\mu_n$ on $\Omega$. These cohomology groups are naturally endowed with actions of $\mathrm{GL}(\mathbb{Q}_p, d+1)$. This point of view is crucial in the local Langlands program.

Next, let us talk about a motivation in complex geometry. Let us consider a nice family of hyperbolic Riemann surfaces over the punctured disk $\Delta^*$. Here nice means admitting a semistable model if this makes sense to you. On each fiber, there is a natural measure called the Bergman measure: take an orthonormal basis $s_1, \dots, s_g \in H^0(X, \omega_{X_t})$ with respect to the obvious inner product, the Bergman measure is just $\beta_t := 2^{-1} \sum_{j=1}^g \mathrm{i} s_j \wedge \overline{s_j}$. It is very natural to wonder how this family of Bergman measures degenerate as $t \to 0$. It turns out that the answer is given by non-Archimedean geometry. Sanal Shivaprasad [Shi24] proved that if we put a non-Archimedean fiber in the middle, then $\beta_t$ converges weakly to a non-Archimedean canonical measure, *Zhang's measure*.

One can summarize the idea as that non-Archimedean objects characterize the degeneration of complex families. This is the point of view I want to emphasize in this series of lectures, addressing to complex geometriers.

As a continuation of this idea, let me explain a third motivation. We will consider the degeneration problem of geodesic rays of Kähler potentials. We will fix a projective manifolds $X$ and a Hodge form $\omega$. It is well-known that the space of regular $\omega$-psh functions has an intrinsic metric geometry. In particular, one can talk about geodesic rays in this space. Studying asymptotics of functionals along these rays is one of the central theme in the variational approach to the Yau–Tian–Donaldson conjecture.

Let us consider such a geodesic ray $\ell_t$. How do we understand the behaviour as $t \to \infty$? It is almost always true that $\ell_t(x) \to -\infty$ for certain $x \in X$. But this is not enough for understanding the degeneration. One needs to know the *speed* of divergence as well in order to get a reasonable understanding of the degeneration.

Here the non-Archimedean picture comes into play again. In this case, the non-Archimedean space consists of a compactification of objects like $t \operatorname{ord}_E$, where $t \in \mathbb{Q}_{>0}$ while $E$ is a prime divisor over $X$. One can view $t \operatorname{ord}_E$ as a valuation of the field $\mathbb{C}(X)$ given by $t$ times the order of vanishing along $E$. One can generate a coupling between the ray $\ell$ and $t \operatorname{ord}_E$, giving the speed of degeneration of $\ell$ along $E$. It turns out that if $\ell$ is decent in a specific sense (maximal if you are familiar with [BBJ21]), then $\ell$ is completely determined by these couplings.

In fact, this approach gives us more: any ray $\ell$ induces a non-Archimedean plurisubharmonic function $\ell^{\mathrm{an}}$. One main result proved in [BBJ21] is that decent rays are in bijection with certain non-Archimedean plurisubharmonic functions. This gives us the motivation to study the non-Archimedean pluripotential theory.

The three examples correspond to the three different choices of the valued field which we talked about earlier.

I hope that now you have enough motivations to begin to learn the non-Archimedean geometry.

## 2. Non-Archimedean pluripotential theory

In this series of lectures, I will address the problem of non-Archimedean pluripotential theory, namely the study of plurisubharmonic functions on non-Archimedean spaces.

Since this is a course addressing to complex geometriers, I will not talk about the traditional approach to the non-Archimedean geometry. Rather, I will content myself to the *ad hoc* approach as in Boucksom–Jonsson's paper.

Now it is a good point to talk about different theories of non-Archimedean geometry. So far, there are three mainstream and well-established approaches to the non-Archimedean geometry: Tate's theory of rigid spaces, Berkovich's theory of Berkovich spaces and Huber's theory of adic spaces. Instead of getting the readers to the complicated comparison, it suffices for us to mention two features that singles out the Berkovich theory: first of all, Berkovich spaces have the best topological properties; secondly, Berkovich theory also makes sense when the base field is trivially valued. Both features will play important roles in the sequel.

Let us make a pause and start with the simplest example of a Berkovich space. The analytification of $\mathbb{P}^1$ over the trivially valued field $\mathbb{C}$. Geometrically, it is mostly intuitive to draw the picture first as in Fig. 1. It might seem striking and counter-intuitive to call such a bizarre object $\mathbb{P}^1$. But you will get used to this picture when you get more familiar with the non-Archimedean geometry.

This picture itself requires some further explanation. There is a center in the graph, called the *Gauss point* or the *trivial valuation*, denoted by $v_{\mathrm{triv}}$. There are infinitely many legs attached to this special point. The legs are labeled by points in $\mathbb{P}^1(\mathbb{C})$. So the whole picture can be viewed as a tree with infinitely many legs. The topology is given by the pro-tree topology. In more concrete terms, a base of neighbourhoods around the Gauss point is given by the following sets: take a open subset on each leg containing the Gauss point such that only finitely many among them are not equal to the whole leg, then take their union.

This kind of picture only works for trivially valued fields. In general, the Berkovich projective line exhibits much more complicated behaviours, which depend on the field a lot. I will come back to this point at the end of the lectures.

Figure 1. Berkovich $\mathbb{P}^1$ over a trivially valued field

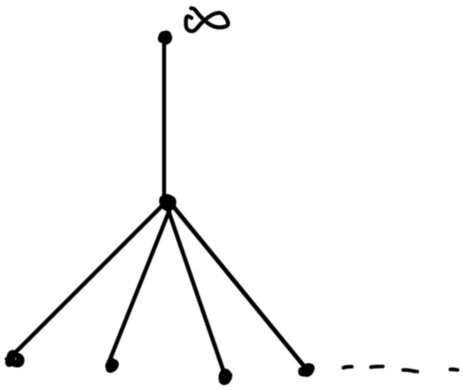


fig:BerP1

In this special picture, we observe that each leg is homeomorphic to $[0, \infty]$ in the obvious way. Here the Gauss point corresponds to the 0-end. There is therefore a scaling action on the pro-tree: $\mathbb{R}_{>0}$ acts on each leg by multiplication. The readers should verify that the scaling is continuous. The fixed points of the scaling action are exactly the Gauss point together with the outer end points of the legs. Again, the existence of the scaling is a unique feature of the trivially valued case.

In the first few lectures, I will only focus on the trivially valued case. This is due to the fact that the general situation, despite its similarity with the trivially valued case, necessities much deeper understandings of the non-Archimedean geometry. I will try to come back to the general situation at the last lecture if we have time.

Now we have a picture of a very simple Berkovich space in mind, I could begin to explain the different approaches to the pluripotential theory on these spaces. So far, there are four different approaches to the pluripotential theory on Berkovich spaces.

(1) The approach of Chambert-Loir–Ducros. This approach works for arbitrary valued fields. In this approach, they first developed the theory of different forms and currents on the Berkovich spaces. This is based on local tropicalizations of Berkovich spaces. Write our Berkovich space as $X$. Roughly speaking, locally we could find morphisms $X \to T$, where $T$ is a split (analytic) torus. The tropicalization of $T$ is a canonical map $T \to \mathrm{Hom}(X^*(T), \mathbb{R}_{>0}^{\dim T}) \cong \mathbb{R}^{\dim T}$. This allows us to define by composition a continuous map $X \to \mathbb{R}^n$. The image of this map is a polytope. One can define the differential (pre-)forms on $X$ by pulling back the Lagenberg forms on polytopes. Of course, allowing $T$ or the local chart of $X$ to vary, we end up with a huge number of forms. A standard shefification procedure allows us to define the differential forms. Then one could easily develop the theory of currents etc. The plurisuhbarmonic functions are defined using the usual curvature condition. As in the usual pluripotential theory, a singular plurisubharmonic is not simply determined by the curvature condition $\mathrm{dd^c} u \geq 0$, it requires some extra assumptions. In the non-Archimedean theory, unfortunately, this approach only works for certain regular plurisubharmonic functions.

The details of this approach are presented in the long article [CLD12]. Ducros mentioned that they are writing a book to expand this paper. We remind the serious readers that there is a known mistake in the original paper [CLD12] as pointed out by Gubler.

Their approach is followed mainly by the Regensburg school.

(2) The most well-known non-Archimedean theory to the complex geometriers is probably the Boucksom–Jonsson theory. This theory also works for arbitrary valued fields, but certain key properties are missing for general fields.

This approach is motivated by S. Zhang's approach to semi-positive metrics. One begins with a polarized complex variety $(X, L)$. One first consider certain semipositive models, or in the trivially valued case, test configurations satisfying some positivity assumption. It is a well-known idea in non-Archimedean due to Raynaud that models give a complete description of the non-Archimedean geometry. In our situation, this idea lead to the notion of model metrics induced by semipositive models. One should regard these model metrics as prototypes of non-Archimedean plurisubharmonic functions. These model metrics are necessarily somewhat regular. In the usual complex pluripotential theory, a general plurisubharmonic metric can be (at least locally) written as decreasing sequences of regular plurisubharmonic metrics. In the non-Archimedean theory, we take this as the definition of general plurisubharmonic metrics.

It turns out that the plurisubharmonic functions defined in this way are well-behaved. But there are a few twists: there is a key conjecture about these metrics known as the envelope conjecture or the continuity of envelopes. This conjecture and its various consequences are at the heart of the pluripotential theory. The conjecture is solved only in a few cases. For example, when $X$ is smooth and the base field is trivially valued. This also explains why we choose to present this case at first.

But even if one solves the envelope conjecture in general, Boucksom–Jonsson's theory is still not the ideal theory. In fact, it is desirable to have a local theory of plurisubharmonic metrics, but Boucksom–Jonsson's theory is global in nature. The same problem persists for the next two theories.

In the next two theories, we only consider the trivially valued base field $\mathbb{C}$.

(3) The third approach is the more recent one developed in my joint paper with Darvas and K. Zhang. It has roots in a series of deep works in complex pluripotential theory in the last decade. In this theory, one defines the non-Archimedean plurisubharmonic functions without referring to the non-Archimedean spaces. In fact, in my series of joint works with Darvas, we established a theory of singularities in complex geometry, which turns out to characterize the non-Archimedean plurisubharmonic functions. This theory requires some deep knowledge of the complex pluripotential theory. Compared to Boucksom–Jonsson's theory, it has two advantages: it works for general compact unibranch Kähler spaces as well; the envelope conjecture trivially holds in our theory. But it has an obvious drawback, due to the lack of a Berkovich analytification, it is not possible to talk about basic objects like the Monge–Ampère operators. But this difficulty is solved more recently by the fourth approach.

(4) The last approach is being developed in the thesis of Pietro Piccione, a student of Boucksom. In his thesis, Piccione introduces a Berkovich space associated with a compact Kähler manifold. Then the pluripotential theory can be developed correspondingly. Since the preprint has not been made public yet, I will not talk about this approach in these lectures.

In these lectures, we will focus on the second approach. The main reference will be [BJ22]. I will give a few lectures about singularities in complex pluripotential theory in Zhejiang university shortly. The details about the third approach will be presented at the end of those lectures.

## 3. The pluripotential theory on $\mathbb{P}^1$

I will begin to talk about the general framework of Boucksom–Jonsson's theory from next time on.

In the remaining part of this lecture, I will explain the potential theory on the trivially valued Berkovich $\mathbb{P}^1$ with respect to the line bundle $L = \mathcal{O}(1)$. The non-trivially valued case is developed in detail in the thesis of Thuillier [Thu05] and in the book of Baker–Rumely [BR10]. I will content myself the the trivially valued field $\mathbb{C}$.

We will write the non-Archimedean $\mathbb{P}^1$ as $\mathbb{P}^{1,\mathrm{an}}$. We will first give a more precise description of Fig. 1. We decompose $\mathbb{P}^{1,\mathrm{an}}$ as two parts:

$$\mathbb{P}^{1,\mathrm{an}} = \mathbb{P}^{1,\mathrm{val}} \amalg \coprod_{x\in\mathbb{P}^1(\mathbb{C})} \{v_{x,\mathrm{triv}}\},$$

where $\mathbb{P}^{1,\mathrm{val}}$ corresponds to the pro-tree with all end points at the $\infty$-side removed. For each $x \in \mathbb{P}^1(\mathbb{C})$, the notation $v_{x,\mathrm{triv}}$ refers to the removed end point on the leg labeled by $x$. For the reason that will be clear in the next lecture, we will write the singleton $\{v_{x,\mathrm{triv}}\}$ as $x^{\mathrm{val}}$. So the previous decomposition can be rewritten as

$$\mathbb{P}^{1,\mathrm{an}} = \mathbb{P}^{1,\mathrm{val}} \amalg \coprod_{x \in \mathbb{P}^1(\mathbb{C})} x^{\mathrm{val}}.$$

This equation exhibits a hierarchy of the Berkovich $\mathbb{P}^1$, a structure which persists in higher dimensions.

The points in $\mathbb{P}^{1,\mathrm{val}}$ can be regarded as valuations of the function field $\mathbb{C}(X)$. To describe them, let us take $x \in \mathbb{P}^1$. Then we have a valuation $\mathrm{ord}_x \colon \mathbb{C}(X)^\times \to \mathbb{Z}$ given by the order of vanishing along $x$. For each $t \in [0,\infty)$, we have a valuation $t\,\mathrm{ord}_x \colon \mathbb{C}(X)^\times \to \mathbb{R}$. The leg labeled by $x$ can be identified with $[0,\infty]$ as we explained earlier. The $[0,\infty)$ part can therefore be interpreted as $t\,\mathrm{ord}_x$. The $t=0$ end can be described as the valuation sending each non-zero element in the function field to 0. This is the so-called trivial valuation $v_{\mathrm{triv}}$.

The $\infty$ ends, by contrast, are not valuations of the field $\mathbb{C}(X)$. They should be regarded as valuations of the field $\mathbb{C}(x) = \mathbb{C}$ instead. We will explain this point of view in the next lecture. Now we have a pretty complete description of the space $\mathbb{P}^{1,\mathrm{an}}$, let us develop the potential theory on it.

**Definition 3.1.** A function $g \colon [0,\infty] \to \mathbb{R} \cup \{-\infty\}$ is $\mathbb{Q}$-*piecewise linear* if there exist rational points $0 = t_0 < t_1 < t_2 < \cdots < t_N$ such that $g(t_i) \in \mathbb{Q}$ for each $i$, $g$ is linear on each interval $[t_i, t_{i+1}]$ ($i = 0, \ldots, N-1$) and $g$ is linear on $[t_N, \infty]$ with rational slope.

In particular, if $g$ takes value in $\mathbb{R}$, then $g$ is necessarily constant on $[t_N, \infty]$.

**Definition 3.2.** A *piecewise linear* function on $\mathbb{P}^{1,\mathrm{an}}$ is a continuous map $f \colon \mathbb{P}^{1,\mathrm{an}} \to \mathbb{R}$ satisfying the following properties:

(1) $f$ is constant on all but finitely many legs;
(2) On each leg, $f$ restricts to a $\mathbb{Q}$-piecewise linear $g \colon [0,\infty] \to \mathbb{R}$.

We write the set of piecewise linear functions on $\mathbb{P}^{1,\mathrm{an}}$ as $\mathrm{PL}(\mathbb{P}^{1,\mathrm{an}})$.

We denote by $\mathrm{PL}^+(\mathbb{P}^{1,\mathrm{an}})$ the subset of $\mathrm{PL}(\mathbb{P}^{1,\mathrm{an}})$ consisting of functions which are furthermore convex on each leg.

**Definition 3.3.** A function in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(\mathcal{O}(1))$ is a function $f \colon \mathbb{P}^{1,\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ satisfying the following properties:

(1) $f$ is constant on all but finitely many legs;
(2) On each leg, $f$ restricts to a $\mathbb{Q}$-piecewise linear $g \colon [0,\infty] \to \mathbb{R} \cup \{-\infty\}$;
(3) $f$ is convex on each leg;
(4) the sum of the slopes at $v_{\mathrm{triv}}$ is no less than $-1$.

A function in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(\mathcal{O}(1))$ taking value in $\mathbb{R}$ is necessarily a piecewise linear function.

A subharmonic function is by definition, the limit of a decreasing net of functions in $\mathcal{H}^{\mathrm{gf}}_{\mathbb{Q}}(\mathcal{O}(1))$. In our case, it admits a more concrete description.

**Definition 3.4.** A subharmonic metric on $\mathcal{O}(1)^{\mathrm{an}}$ is a function $f \colon \mathbb{P}^{1,\mathrm{an}} \to \mathbb{R} \cup \{-\infty\}$ satisfying the following properties:

(1) $f$ is constant on all but finitely many legs;
(2) $f$ is convex on each leg;
(3) the sum of the slopes at $v_{\mathrm{triv}}$ is no less than $-1$.

The set of subharmonic metrics on $\mathcal{O}(1)^{\mathrm{an}}$ is denoted by $\mathrm{SH}^{\mathrm{NA}}(\mathbb{P}^1, \mathcal{O}(1))$.

Given $f \in \mathrm{SH}^{\mathrm{NA}}(\mathbb{P}^1, \mathcal{O}(1))$, we define its Monge–Ampère measure as the measure on $\mathbb{P}^{1,\mathrm{an}}$ consisting of the sum of two parts: the first part is the Laplacian of $f$ restricted to the interior of each leg. The second part is the Dirac mass at $v_{\mathrm{triv}}$ with coefficient given by 1 plus the sum of the slopes at $v_{\mathrm{triv}}$.

As well will see, the description of $\mathbb{P}^1$ and the description of subharmonic functions both have generalizations in higher dimensions.

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